MATRICES WITH NON-NEGATIVE ELEMENTS

THESIS

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INTRODUCTION.

This thesis is presented in support of my candidature for the degree of Ph.D. It contains work carried out at Edinburgh University between October 1950 and May 1952, under the supervision of Professor A.C. Aitken.

I became interested in matrices with non-negative elements when I realized that many of the most interesting algebraic properties of these matrices could be deduced when it was known which elements were positive and which were zero, the actual value of the elements being immaterial. I determined to pursue an investigation to see how far one could proceed along those lines. Though much was known about such methods, much remained, and still remains, to be done.

I had planned to give a full account of all methods used in deriving the properties of matrices with non-negative elements, and then to add some chapters of original work. It was only when I had written up a large part of my notes that I realized that this plan would make the thesis far too long. Consequently both parts of my material have had to be cut considerably. For example, I had intended to use the work of Chapter 5 to develop more fully a method used by A. Ostrowski (1937) to prove the fundamental properties of matrices with non-negative elements, making use of some results in the theory of the complex variable. There was also to have been a chapter on the iteration with a square matrix $A = [a_{ij}]$, $a_{ij} \geq 0$, a subject which has been studied intensively when A is a

" P - matrix " (Cf. 8.13) but very little otherwise. Some interesting theorems, not applicable to the iteration with a general matrix A, may be obtained here.

My thanks are due to my wife for help in preparing this thesis, to members of the staff and research students in the Department of Mathematics at Edinburgh University for many stimulating discussions, and particularly to Professor Aitken.

SUMMARY OF CONTENTS.

Chapter 1 is a short introductory chapter dealing with some definitions and basic properties of matrices and vectors.

In Chapter 2 we introduce a partial order between matrices with real elements. For matrices with non-negative elements our notation and terminology differ from the usual ones. The casual reader is advised to read section 2.3 before glancing further. The term "Positive Matrix" will be used from now on in the sense of 2.3.

In Chapter 3 we consider the "normal form" of a reducible matrix, and some associated sets. We define the " R - functions," a principal tool of investigation in later chapters.

Chapter 4 contains some consequences of the partial order between matrices. Some analogues of the properties of positive matrices and positive numbers are developed.

In Chapter 5 we consider "chains of elements" and powers of positive matrices.

Chapter 6 contains a resumé of the chief algebraic properties of a matrix that are required in the rest of the thesis. These concern latent roots and latent vectors, sets of "generalized latent vectors", classical canonical sub-matrices, and principal idempotent and nilpotent elements.

In Chapter 7 we describe a method of proving the fundamental properties of positive matrices, which is based on some work by Frobenius.

In Chapter 8 we review a method due to Wielandt (1950) of proving the basic results for irreducible positive matrices. We give a variant of our own. Lower and upper bounds are found, in terms of the elements of the matrix, for the ratios of elements of the strictly positive latent column vector associated with the largest positive latent root of an irreducible positive matrix.

In Chapter 9 we deal with " P-matrices ". We deduce a large number of algebraic results purely by inspection of positive elements. Finally we examine "sets" of latent row vectors.

Chapter 10 is the longest chapter. In it we consider the singular matrix $A = \rho I - P$, where P is a positive matrix and ρ its largest positive latent root. We examine the number of linearly independent latent vectors associated with the latent root 0 (10.9), sets of positive generalized latent vectors (10.16), and, when the multiplicity of 0 does not exceed three, the classical canonical submatrices associated with 0. Provided we know which A_{ii} are singular, when A is in normal form, these questions may be answered by an inspection of the positions of non-zero elements. In general the orders of the classical canonical submatrices associated with 0 can not be settled in this way, though such methods suffice to determine whether they are all equal to 1, (10. 31). Finally we consider the principal idempotent and nilpotent elements of A associated with 0, in some special cases.

The Bibliography then follows. In the text we give as reference the author's name and the date of publication, thus: Frobenius (1912).

The Appendix consists of a paper accepted by the Journal of the London Mathematical Society, "An inequality for latent roots applied to determinants with dominant principal diagonal". The theory of matrices with dominant principal diagonal is closely connected with that for positive matrices. In this paper we "rejected" notation and terminology of 2.3.

CHAPTER I.

1. An $n \times m$ matrix A is an ordered array of elements

$$A : [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & & & & & & \\ \cdot & \cdot & & & & & & \\ \cdot & \cdot & & & & & & \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{nm} \end{bmatrix}.$$

If $n \neq m$ the matrix is called rectangular. If $n = m$, then A is called a square matrix of order n.

When $n = 1$, the matrix

$$u' = [u_1, \ u_2, \ \ldots, u_m]$$

is called a "row vector".

When $m = 1$, the matrix

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix},$$

is called a column vector, and may conveniently be written

$$v = \{v_1 , v_2, \ldots . v_n\}.$$

The row vector $[a_{r1} , \ . \ . \ . \ . \ a_{rm}]$, where a_{rj} $j = 1, \ldots . m$ are elements of a matrix A, is called the r-th row of A, and will be denoted by a_{r*}. Similarly, a_{*r} will denote $\{a_{1r}, \ a_{2r} \ . \ . \ . \ . \ a_{nr}\}$, and will be called the r-th column of A. The $n \times m$ matrix A is thus said to have n

rows and m columns.

The elements a_{ij} of A may lie in any algebraic system. We shall however be concerned only with elements in the real or complex field.

It will be seen that our definitions of matrix and vectors are rather old-fashioned. We shall be investigating in detail the structure of a type of matrix and some associated vectors. A more general axiomatic definition of vector, etc., would therefore have no advantage for our purposes.

2. Matrices having the same number of rows, and of columns, will be said to be "of similar shape".

The addition of matrices is defined for matrices of similar shape:

If $A : [a_{ij}]$, and $B : [b_{ij}]$ are of similar shape, then

$$A + B = [a_{ij} + b_{ij}]$$

The product AB is defined when B has as many rows as A has columns. Matrices A, B, whose product AB is defined are said to be conformable. (It may sometimes be left to the context to indicate whether AB or BA is defined).

If $A : [a_{ij}]$ is an $n \times r$ matrix, and $B : b_{ij}$ is an $r \times m$ matrix, then AB is the $n \times m$ matrix

$$AB = \left[\sum_{k=1}^{r} a_{ik} \; b_{kj} \right] .$$

The set of all square matrices of order n with complex elements form a non-commutative ring. (Cf. Birkhoff and

MacLane (1941) p. 348).

For $A + B$, and AB exist,

and $$A + B = B + A$$

$$A(B + C) = AB + AC$$

$$A(BC) = (AB)C$$

as may easily be verified.

The set of all square matrices of order n with real elements also form a non-commutative ring. To see this it is only necessary to note, that in addition to the above properties, the sum or product of matrices of real elements also has real elements.

Let $$\sigma_{ij} = 1 \quad \text{when} \quad i = j,$$
$$\sigma_{ij} = 0 \quad \text{when} \quad i \neq j.$$

The square matrix $[\sigma_{ij}]$, $i, j = 1, \dots n$ is denoted by I. When A, I, and I, B are conformable with I we have

$$A I = A, \qquad I B = B.$$

The matrix $[a_{ij}]$, $a_{ij} = 0$ $i = 1 \dots . n$, $j = 1 \dots . m$ will be denoted by 0. When A is of similar shape,

$$A + 0 = 0 + A = A,$$

while $$A 0 = 0, \quad 0 B = 0,$$

when A and 0, 0 and B are conformable.

It is usually unnecessary to indicate the shape of I and 0 in any particular case. The shape will generally be clear from the context in which I and 0 occur.

A square matrix A whose determinant vanishes is called "singular". Otherwise A is said to be non-singular. There is a matrix A satisfying $A A^{-1} = I$ if and only if A is non-singular. When A is non-singular the matrix A^{-1} satisfying $A A^{-1} = I$ also satisfies $A^{-1} A = I$. If $B A = I$, or $A C = I$, then $B = C = A^{-1}$. There is a column vector x conformable with A satisfying $Ax = 0$ if and only if A is singular. (For a definition of the determinant of A, and proofs of the above assertions see Aitken (1939))

CHAPTER 2.

2.1

In this chapter we shall be concerned with introducing a partial order between matrices of real elements. Such matrices will be called real matrices.

Definitions. Let $A : [a_{ij}]$, $B : [b_{ij}]$ be real matrices of similar shape, say $n \times m$.
If $a_{ij} \geqslant b_{ij}$, $i = 1, \ldots n$, $j = 1, \ldots m$, we shall write $A \geqq B$.
If $A \geqq B$, but $A \neq B$ we shall write $A \supsetneqq B$.
It is easily seen from these definitions that,

$$A \geqq A.$$

If $A \geqq B$, and $B \geqq A$, then $A = B$,
If $A \geqq B$, and $B \geqq C$, then $A \geqq C$,
A, B, C, being matrices of similar shape.

These are the requirements for the set of matrices to form a partially ordered system. (Cf. Birkhoff and MacLane (1941) p. 326).

We shall write $A > B$, when $a_{ij} > b_{ij}$, $i = 1, 2 \ldots n$, $j = 1, 2 \ldots m$, and $A \geqslant B$ when either $A > B$ or $A = B$.
It is easily seen that the set of real matrices is partially ordered also under "$\geqslant$". When neither $A \geqslant B$ nor $B \leqslant A$, we shall write $A \| B$.

The relation "$A \geqq B$" may also be written as "$B \leqq A$", "$A \supsetneqq B$" as "$B \subsetneqq A$", "$A > B$" as "$B < A$" etc.

2.2

Let E, F, be the sets of integers (1, 2, ... n), (1, 2, ... m) respectively, and let G, H, be subsets of E and F. We shall denote this relation by $G \prec E$, $H \prec F$ or $E \succ G$, $F \succ H$. The matrix $M : [a_{ij}]$ $i \in G$, $j \in F$, will be called a submatrix of $A : [a_{ij}]$ $i \in E$, $j \in F$. The matrix $N : [b_{ij}]$, $i \prec G$, $j \prec F$, is a submatrix of $B : [b_{ij}]$ $i \in E$, $j \in F$ and will be called "the submatrix of B corresponding to M".

<u>Theorem. Let A and B be matrices of similar shape. Then $A > B$ if and only if $M \supset N$ for all $M \prec A$ and corresponding $N \prec B$.</u>

Let $M \supset N$, for all corresponding M, N. The 1 x 1 matrices $[a_{ij}]$, $[b_{ij}]$ correspond;

hence $a_{ij} > b_{ij}$, $i \in E$, $j \in F$,

and so $A > B$.

If $A > B$ and $M : [a_{ij}]$, $N : [b_{ij}]$, $i \in G$, $j \in H$,

then, as $a_{ij} \geq b_{ij}$, $i \in G$, $j \in H$,

we have $M \supseteq N$.

But $M \neq N$, as $a_{ij} > b_{ij}$, $i \in G$, $j \in H$.

Hence $M \supset N$.

The theorem indicates the connection between the relations " $\supset$ " and " $>$ " , and might have been used to define "$A > B$".

2.3

We may read "$A \geq B$" as "A is (weakly) greater, or equal to B".

We may read:

"$A > B$" as "A is (weakly) greater than B" .

"$A \geq B$" as "A is strictly greater or equal to B".

"$A \gg B$" as "A is strictly greater than B".

"$A \parallel B$" as "A and B are incomparable".

We may also read:

"$B \leq A$" as "B is (weakly) smaller than A".

"$A > 0$" as "A is (weakly) positive".

"$A \ll 0$" as "A is strictly negative".

"$A \leqq 0$" as "A is (weakly) negative or zero".

When "greater", "positive" etc. are not qualified it is understood that the weak relations are intended.

It will be noticed that in the case of positive matrices we have slightly departed from a notation and terminology that has been used by several writers, e.g. Frobenius (1908, 1912), Ostrowski (1937), Wielandt (1950), Ledermann (1950 a).

We shall give a summary of these conventions.

Rejected Notation.

Let A be an n x m matrix.

When $a_{ij} > 0$, $i = 1, \dots n$, $j = 1, \dots m$,

then $A > 0$, and "A is positive".

When $a_{ij} \geq 0$, $i = 1, \dots n$, $j = 1, \dots m$,

then $A \geq 0$, and "A is non-negative".

Our notation is considerably more convenient and rather more satisfactory logically. For example, we shall find it essential to distinguish between weakly positive and zero matrices. In our notation it is sufficient to write "$A > 0$". In the rejected notation we should have to write "$A \gtrsim 0$, $A \neq 0$". This is clumsy, and if read "A is non-negative, but non-zero", then "non-negative" means "no element is negative", while "non-zero" means "some element is not zero". In the rejected notation "$A \geqslant 0$" does not imply "either $A > 0$ or $A = 0$", and the partial order under both "$\geq$" and "$\gtrsim$" (in our notation) is not brought out. From the practical point of view the relation "$A \geqslant B$" in our notation may seem rather artificial, but we shall have occasion to use it. In the rejected notation we should have to write "$A > B$ or $A = B$"; the symbol "$\geqslant$" seems natural.

We shall not apply the terms "non-negative", "non-positive" to matrices. In the case of scalars we shall follow the usual conventions.

If 1×1 matrices $A : [a_{11}]$ are identified with the scalars "a" then the symbols "$>$", "$\geqslant$", etc., retain their usual meanings, and there is in this case no difference between weak and strong inequalities.

CHAPTER 3.

3.1.

Let E be the set of integers 1, 2, ... n.
Let A be the square matrix of order n, $A : [a_{ij}]$, $i, j \in E$.
Let G be a non-empty subset of E such that E - G, the complement of G in E, is also non-empty. Under these conditions G is called a proper subset of E.
Let M be the submatrix of A, $M : [a_{ij}]$, $i \in G$, $j \in E - G$.
The matrix A is said to be reducible if there is some proper subset G such that $M = 0$.

If A is not reducible it is said to be irreducible.
A conjugate permutation of rows and columns is one which permutes the rows and columns indices in the same manner, i.e., there is a permutation $(h_1, h_2, \ldots h_n)$ of $(1, 2, \ldots n)$ such that $a_{ij} = \bar{a}_{h_i h_j}$; $\bar{A} : [\bar{a}_{ij}]$, being the matrix obtained after the conjugate permutation.

It should be noted that the reducibility of a matrix is not affected by a conjugate permutation of rows and columns.

For suppose there is a proper subset G of E, such that

$$a_{ij} = 0, \quad \text{when} \quad i \in G, \quad j \in E - G.$$

Let $h_i \in \bar{G}$ when $i \in G$.

Then $\bar{a}_{h_i h_j} = a_{ij} = 0$ when $h_i \in \bar{G}$, $h_j \in E - \bar{G}$, and so $\bar{A}$ is reducible. Similarly, we may prove that A is reducible when $\bar{A}$ is reducible.

We shall adopt a usual convention for the union intersection of classes :

$i \in E_1 \cup E_2$ if and only if either $i \in E_1$, or $i \in E_2$,

$i \in E_1 \cap E_2$ if and only if $i \in E_1$ and $i \in E_2$.

3.2 Let $E_1, E_2, \ldots, E_k$ be sets satisfying

$$E_\alpha \neq 0 , \qquad \alpha = 1 \ldots k \quad \text{(0 is the empty set)}; \qquad \text{(a)}$$

$$\bigcup_{\alpha=1}^{k} E_\alpha = E = (1, 2 \ldots n), \; \left(\bigcup_{\alpha=1}^{k} E_\alpha = E_1 \cup E_2 \cup \ldots \cup E_k\right) ; \qquad \text{(b)}$$

$$E_\alpha \cap E_\beta = 0, \qquad \alpha \neq \beta . \qquad \text{(c)}$$

Theorem 1. There exist non-empty sets E_i, $i = 1 \ldots k$, satisfying (a) (b) (c) of 3.2 , for which

(i) $A_{\alpha\alpha} : [a_{ij}]$, $i, j \in E_\alpha$, is irreducible ,

(ii) $A_{\alpha\beta} = [a_{ij}] = 0$, $i \in E_\alpha$, $j \in E_\beta$, when $\alpha < \beta$.

Note: The matrices $A_{\alpha\alpha}$, $A_{\alpha\beta}$, $\beta \neq \alpha$, are not completely defined, as the order of the $i \in E_\alpha$ is left undetermined. Hence $A_{\alpha\alpha}$ is determined up to a conjugate permutation of rows and columns. By 3.1 such a permutation leaves $A_{\alpha\alpha}$ irreducible. The matrix $A_{\alpha\beta}$, $\beta \neq \alpha$, is determined up to some permutation of rows and columns. Of course, a null-matrix remains null after any permutation of rows and columns.

The proof of the theorem, is by induction on n, the order of the matrix. The theorem is trivial when $n = 1$; $E_1 = E$ is the only non-empty subset of E.

Suppose the theorem is true for $n = 1, 2, \ldots\ldots r$, and let $n = r + 1$. If A is irreducible, the theorem is clearly satisfied for $k = 1$.

If A is reducible, there is a proper subset $G \prec E$ such that

$$M = 0 \ , \ M : [a_{ij}] \quad i \in G, \ j \in E - G .$$

Now let A_1 be $A_1 : [a_{ij}] \quad i, j \in G$,
and let $A_2 \prec A$ be, $[A_2]: \ [a_{ij}] \quad i, j \in E - G$.

By the inductive hypothesis there are sets

$E_1, E_2, \ldots\ldots, E_l \ (E_{l+1} \ldots\ldots\ldots E_k)$ such that

$$E_\alpha \cap E_\beta = 0 \ , \quad 1 \leq \alpha, \beta \leq l, \ (l+1 \leq \alpha, \beta \leq k), \ \alpha \neq \beta,$$
$$\bigcup_{\alpha=1}^{l} E_\alpha = G \quad , \left(\bigcup_{\alpha=l+1}^{k} E_\alpha = E - G \right),$$

and (i) and (ii) of theorem 1 are satisfied for $\{ 1 \leq \alpha, \beta \leq l, \ (l+1 \leq \alpha, \beta \leq k) \}$. Evidently the sets E_α, $\alpha = 1, \ldots, k$ satisfy (a), (b), (c) of 3.2 and theorem 1, (i).

Also $A_{\alpha\beta} \prec M = 0$, $1 \leq \alpha \leq l$, $l + 1 \leq \beta \leq k$,
whence $A_{\alpha\beta} = 0$, $\alpha < \beta$, $1 \leq \alpha, \beta \leq k$,
by theorem 1 (ii) applied to $1 \leq \alpha, \beta \leq l$, and to $l+1 \leq \alpha, \beta \leq k$.
This proves the theorem.

3.3 The R-function;

Let E_α, $\alpha = 1, \ldots\ldots k$ be sets satisfying 3.2 (a), (b), (c), and let $A_{\alpha\alpha} : [a_{ij}] \quad i, j \in E_\alpha$, be irreducible submatrices of A .

Let $A_{\alpha\beta}$: $[a_{ij}]$ $i \in E_{\alpha}$, $j \in E_{\beta}$.

Define $r_{\alpha\beta} > 0$, if $A_{\alpha\beta} \neq 0$ $\alpha \neq \beta$,

$r_{\alpha\beta} = 0$, if $A_{\alpha\beta} = 0$, $\alpha \neq \beta$.

Let $R_{\alpha\alpha} = 1$, $\alpha = 1, \ldots, k$,

and $R_{\alpha\beta} = \sum r_{k_1k_2} r_{k_2k_3} \ldots r_{k_{s-1}k_s}$, $\alpha \neq \beta$,

where $k_1 = \alpha$, $k_s = \beta$, and the summation is taken over all possible distinct ~~sets~~ sequences $(k_1, k_2, \ldots . k_s)$ with $k_i \neq k_{i+1}$. This definition of $R_{\alpha\beta}$ will often imply that $R_{\alpha\beta}$ is infinite, ~~but~~ As we shall never be concerned with the value of $R_{\alpha\beta}$, but only wish to distinguish $R_{\alpha\beta} > 0$ from $R_{\alpha\beta} = 0$, this does not matter. In any case, ~~in the case~~ when the E_{α} also satisfy 3.2 Theorem 1 (ii) the $R_{\alpha\beta}$ are finite. When the E_{α} satisfy 3.2 (a), (b), (c), and theorem 1, (i) and (ii) we shall call them the "decomposition sets" of A. In chapters 9 and 10 we shall deduce interesting applications of the R-functions.

Lemma. If the E_{α} are decomposition sets of A , then $R_{\alpha\beta} = \sum r_{k_1k_2} r_{k_2k_3} \ldots r_{k_{s-1}k_s}$, $k_1 = \alpha$, $k_s = \beta$, summed over the $2^{\beta-\alpha-1}$ terms satisfying $k_1 > k_2 > \ldots > k_s$

$\alpha-\beta-1$

If the E_{α} are decomposition sets of A, then $A_{\alpha\beta} = 0$, $\alpha < \beta$.

Hence $r_{\alpha\beta} = 0$, where $\alpha < \beta$.

Every term $r_{k_1k_2} \ldots r_{k_{s-1}k_s}$ of $R_{\alpha\beta}$ containing a factor $r_{k_ik_{i+1}}$ $k_i < k_{i+1}$, therefore, vanishes. It follows that $R_{\alpha\beta} = \sum r_{k_1k_2} \ldots r_{k_{s-1}k_s}$

summed over the terms indicated in the Lemma.

3.4.

Theorem 2. Let E_α, $\alpha = 1, \ldots, k$, satisfy the conditions 3.2 (a), (b), (c), and theorem 1, (i). Then the E_α satisfy 3.2 theorem 1 (ii), (i.e. are decomposition sets of A) if and only if

$$R_{\alpha\beta} = 0 \qquad \text{when } \alpha < \beta .$$

If the E_α satisfy the conditions of theorem 1 (are decomposition sets) then by 3.3 Lemma,

$$R_{\alpha\beta} = \sum r_{k_1 k_2} \ldots r_{k_{s-1} k_s}, \quad k_1 = \alpha, \quad k_s = \beta,$$

summed over those sequences k_i which satisfy $k_1 > k_2 > \ldots > k_s$. When $\alpha < \beta$ there is no such sequence, and hence $R_{\alpha\beta} = 0$.
Conversely let $R_{\alpha\beta} = 0$ when $\alpha < \beta$.
Then as all terms of $R_{\alpha\beta}$ are non-negative, and $r_{\alpha\beta}$ is a term of $R_{\alpha\beta}$ it follows that

$$r_{\alpha\beta} = 0,$$

and so also $\qquad A_{\alpha\beta} = 0.$

Hence the E_α satisfy (ii) of theorem 1, and consequently are decomposition sets of A.

3.5

Theorem 3. Let E_α, $\alpha = 1, \ldots, k$ and E'_α, $\alpha = 1, \ldots, l$, be sets satisfying 3.2 (a), (b), (c). If the E_α satisfy the conditions of theorem 1 for a matrix A then the E'_α satisfy that theorem if and only if

(i) $l = k$;

(ii) There is a permutation $(h_1, \ldots h_k)$ of the integers $(1, \ldots k)$ such that $E'_\alpha = E_{h_\alpha}, \alpha = 1, \ldots k$,

(iii) $R_{h_\alpha h_\beta} = 0$ when $\alpha < \beta$.

Let $A'_{\alpha\beta}$ be the matrix $A'_{\alpha\beta} : [a_{ij}]$, $i \in E'_\alpha$, $j \in E'_\beta$; and let $r'_{\alpha\beta}$, $R'_{\alpha\beta}$ be defined for the E'_α as $r_{\alpha\beta}$, $R_{\alpha\beta}$ for the E_α .

If (i) and (ii) hold then $A'_{\alpha\alpha} = A_{h_\alpha h_\alpha}$ is square and irreducible. If (iii) holds, $r_{h_\alpha h_\beta} = 0$ when $\alpha < \beta$. Hence $A'_{\alpha\beta} = A_{h_\alpha h_\beta} = 0$. This proves that the E'_α satisfy theorem 1.

Now suppose that the E'_α satisfy the conditions of theorem 1. Let γ be the smallest integer such that $E_\alpha \cap E'_\gamma = G \neq 0$. If $G \neq E_\alpha$ then $M = 0$, where $M : [a_{ij}]$, $i \in G \prec E_\alpha$, $j \in E_\alpha - G$, and thus $A_{\alpha\alpha}$ is reducible, contrary to the conditions of theorem 1. Hence $G = E_\alpha$, whence $E'_\gamma \succ E_\alpha$. Similarly $E_\alpha \supset E'_\gamma$. Hence $E_\alpha = E'_\gamma$.

But $E'_\alpha \neq E'_\beta$, $\alpha \neq \beta$; and so there is one E'_γ corresponding to each E_α .

We immediately deduce (i) and (ii).

If $E'_\alpha = E_{h_\alpha}$, $A'_{\alpha\beta} = A_{h_\alpha h_\beta}$ and $r'_{\alpha\beta} = r_{h_\alpha h_\beta}$. It follows that $R'_{\alpha\beta} = R_{h_\alpha h_\beta}$.

But under the conditions of theorem 1, $R'_{\alpha\beta} = 0$ if $\alpha < \beta$, and hence $R_{h_\alpha h_\beta} = 0$, when $\alpha < \beta$.

3.6.

Suppose $\alpha > \beta$ and $R_{\alpha\beta} = 0$.

$$R_{\alpha\beta} = R_{\alpha\gamma} R_{\gamma\beta} + \text{(non-negative terms)}.$$

Hence $R_{\alpha\gamma} R_{\gamma\alpha} = 0$.

There are three possibilities for any γ.

(i) $R_{\alpha\gamma} > 0$, $R_{\gamma\beta} = 0$.

Let $\gamma \in X_1$, $\alpha \geq \gamma \geq \beta$, if and only if γ satisfies (i).

Suppose X_1 has k_1 members;

(ii) $R_{\alpha\gamma} = 0$, $R_{\gamma\beta} = 0$.

Let $\gamma \in X_2$, $\alpha \geq \gamma \geq \beta$, if and only if γ satisfies (ii).

Suppose X_2 has k_2 members;

(iii) $R_{\alpha\gamma} = 0$, $R_{\gamma\beta} > 0$.

Let $\gamma \in X_3$, $\alpha \geq \gamma \geq \beta$, if and only if γ satisfies (iii).

Suppose X_3 has k_3 members.

Clearly $\alpha - \beta + 1 = k_1 + k_2 + k_3$.

<u>Lemma 1.</u> <u>Let $\alpha \geq i, j \geq \beta$;</u>

<u>If $i \in X_\mu$, $j \in X_\nu$ and $\mu < \nu$ ($\mu, \nu = 1, 2, 3$),</u>

<u>then $R_{ij} = 0$.</u>

Suppose $i \in X_1$, $j \in X_2$ or $j \in X_3$.

Then $0 = R_{\alpha j} = R_{\alpha i} \cdot R_{ij}$ + (non-negative terms).

Whence $R_{\alpha i} \cdot R_{ij} = 0$.

But $R_{\alpha i} > 0$, as $i \in X_1$.

It follows that $R_{ij} = 0$.

Suppose $i \in X_2$, $j \in X_3$.

We consider $0 = R_{i\beta} = R_{ij} R_{j\beta}$ + (non-negative terms)

But $R_{j\beta} > 0$, whence $R_{ij} = 0$.

This proves the lemma.

Theorem 4. Let E_α, $\alpha = 1, \ldots, k$ be sets satisfying ~~the~~ (a), (b), (c) of (3.2.) and the conditions of theorem 1. (The E_α are decomposition sets of A). There is an arrangement $E'_{h_\alpha} = E_\alpha$, $\alpha = 1, \ldots k$ of the E_α satisfying theorem 1 such that $h_\alpha < h_\beta$ if and only if $R_{\alpha\beta} = 0$; (the set $(h_1 \ldots h_k)$ is a permutation of $(1, 2, \ldots k)$).

The condition is necessary by (iii) of theorem 3.

Suppose $R_{\alpha\beta} = 0$.

If $\alpha < \beta$ the E_α are a required arrangement.

If $\alpha = \beta$, then $R_{\alpha\beta} = 1$ and the condition does not apply.

Suppose $\alpha > \beta$. Let us define the sets X_1, X_2, X_3, as at the beginning of the section.

Put $h_\gamma = \gamma$, $1 \leq \gamma \leq \beta$,

$h_\gamma = \beta - 1 + i$ where $\gamma \in X_1$, and $i < i'$ when $\gamma < \gamma'$,

$h_\gamma = \beta - 1 + k_1 + i$ where $\gamma \in X_2$ and $i < i'$ when $\gamma < \gamma'$,

$h_\gamma = \beta - 1 + k_1 + k_2 + i$ where $\gamma \in X_3$ and $i < i'$ when $\gamma < \gamma'$,

$h_\gamma = \beta - 1 + k_1 + k_2 + k_3 + i = \gamma$, $\alpha < \gamma \leq k$, and $i < i'$, when $\gamma < \gamma'$.

The E'_γ satisfy theorem 1. For theorem 1 (i) is clearly satisfied as $A'_{h_\gamma h_\gamma} = A_{\gamma\gamma}$. We may check that $R'_{h_\gamma h_\delta} = 0$ when $h_\gamma < h_\delta$ ($R'_{\alpha\beta}$ being defined for the E'_α as $R_{\alpha\beta}$ for the E_α) by inspecting each of the five sets in which the h_γ are grouped above.

Suppose, for example, that $\gamma \in X_2$, and that $h_\gamma < h_\delta$.
If $\delta \in X_2$, $\delta > \gamma$ by construction. Thus $R'_{h_\gamma h_\delta} = R_{\gamma\delta} = 0$.
If $\delta \in X_3$, $R'_{h_\gamma h_\delta} = R_{\gamma\delta} = 0$, by the Lemma.

If $\delta \notin X_2$ $\delta \notin X_3$, then $h_\delta > \alpha$,

whence $\delta > \alpha$, while $\gamma < \alpha$.

Hence $R'_{h_\gamma h_\delta} = R_{\gamma\delta} = 0$

~~The theorem follows~~. Thus the E'_α satisfy Theorem 1, (ii).

Now $\alpha \in X_1$ and if $\gamma \in X_1$, then $\gamma < \alpha$.
Hence it follows from our construction that,
if $h_\alpha = \beta - 1 + j$ and $h_\gamma = \beta - 1 + i$,
then $j > i$.
Hence $h_\alpha = \beta - 1 + k_1$.
By a similar argument we obtain from $\beta \in X_3$ that

$$h_\beta = \beta + k_1 + k_2 .$$

Consequently $h_\alpha < h_\beta$,
the required relation.

3.7.

It is possible to express the results of this chapter in a different way.

Let us suppose the decomposition sets of A, E_α, (c.f. 3.3) have n_α members, $\alpha = 1......k$. We may carry out a conjugate permutation of rows and columns on A such that

$$0 < h_i \leq n_1 \quad \text{when} \quad i \in E_1,$$
$$n_1 < h_i \leq n_1 + n_2 \quad \text{when} \quad i \in E_2,$$
$$\sum_{\alpha=1}^{k-1} n_\alpha < h_i \leq \sum_{\alpha=1}^{k} n_\alpha = n \quad \text{when} \quad i \in E_k .$$

If $\bar{A} : [\bar{a}_{ij}]$, $a_{ij} = \bar{a}_{h_i h_j}$, $i, j = 1.....k$,

$$\text{Then } \bar{A} = \begin{bmatrix} \bar{A}_{11} & & & \\ \bar{A}_{21} & \bar{A}_{22} & & \\ \cdot & & \cdot & \\ & & & \cdot \\ \bar{A}_{k1} & \bar{A}_{k2} & \cdots & \bar{A}_{kk} \end{bmatrix},$$

where $\bar{A}_{\alpha\alpha}$, $\alpha = 1, \ldots, k$, is irreducible.

For $\bar{A}_{\alpha\beta}$ is derived from $A_{\alpha\beta}$ of theorem 1, by a permutation of rows and columns.

Hence $\bar{A}_{\alpha\beta} = 0$, when $\alpha < \beta$,

as $\tilde{A}_{\alpha\beta} = 0$, when $\alpha < \beta$.

The matrix $\bar{A}_{\alpha\alpha}$ is irreducible, as $A_{\alpha\alpha}$ is irreducible by 3.1. A matrix of the form of $\bar{A}$ (i.e. such that the decomposition sets are $(1, 2, \ldots\ldots n_1), (n_1 + 1, \ldots\ldots n_1 + n_2)$, etc.), will be said to be in "normal" form. The $\bar{A}_{\alpha\beta}$, $\alpha, \beta = 1, \ldots, k$, will be called the "decomposition sub-matrices of A".

3.8.

The R-condition may be stated in terms of "chains" whose members are the $A_{\alpha\beta}$, $\alpha \neq \beta$. Let us call $(\alpha_1, \alpha_2 \ldots \alpha_s)$ a "chain" when $\alpha_i \neq \alpha_{i+1}$ and $A_{\alpha_i \alpha_{i+1}} \neq 0$, $i = 1, 2, 3, \ldots s - 1$. The length of the chain is defined as one less than the number of members of the chain.

Theorem 5. Let $\alpha \neq \beta$.

$R_{\alpha\beta} > 0$, if and only if there is chain $(\alpha, \ldots \beta)$

There is a chain $(\alpha, \ldots, \beta)$ if and only if there are integers $\alpha_1, \alpha_2, \ldots, \alpha_s$ satisfying

(i) $\alpha_1 = \alpha$, $\alpha_s = \beta$;

(ii) $\alpha_i \neq \alpha_{i+1}$, $i = 1, 2, \ldots\ldots, s - 1$,

(iii) $A_{\alpha_i \alpha_{i+1}} \neq 0$, $\quad i = 1, 2, \dots, s - 1$.

Hence by the definitions of 3.3 there is a chain $(\alpha, \dots, \beta)$ if and only if there are integers satisfying (i), (ii) and

(iv) $r_{\alpha_1 \alpha_2} \cdot r_{\alpha_2 \alpha_3} \cdots r_{\alpha_{s-1} \alpha_s} = 1$.

But $R_{\alpha\beta} > 0$ if and only if there are integers $\alpha_1, \alpha_2, \dots, \alpha_s$ satisfying (i), (ii) and (iv). ~~(Not every term of R need satisfy (ii) but of course we may omit factors R in the products as = 1)~~.

Hence $R_{\alpha\beta} > 0$, if and only if there is a chain $(\alpha, \dots, \beta)$.

3.9.

In view of the results of 3.7. and 3.8. it is possible to enunciate theorems 2, 3, 4, rather differently, e.g. theorem 4 may be restated thus:

Theorem 4 a. Let A be in normal form, and let $A_{\alpha\beta}$, $\alpha, \beta = 1, \dots, k$ be the decomposition sets of A.

Then there is a permutation $(h_1, \dots, h_k)$ of $(1, 2, \dots, k)$ such that both $\bar{A}$ is in normal form,

where $\bar{A} : [\bar{A}_{\alpha\beta}]$, and $\bar{A}_{h_\alpha h_\beta} = A_{\alpha\beta}$, $\alpha, \beta = 1, \dots, k$,

and $h_\gamma < h_\delta$,

if and only if there is no chain $(\gamma, \dots, \delta)$.

By 3.7. we may replace "E_α, E'_α are decomposition sets", by "A, $\bar{A}$ are in normal form and $A_{\alpha\beta}$, $\bar{A}_{\alpha\beta}$ are decomposition submatrices".

By theorem 5, we may replace "$R_{\alpha\beta} = 0$" by "there is no chain $(\alpha, \ldots, \beta)$".

The theorem then follows.

CHAPTER 4.

4.1. Some Properties of Positive Matrices.

It is not here our purpose to develop fully the consequences of the partial order introduced between real matrices of similar shape in chapter 2. We shall content ourselves by stating explicitly those relations we shall later use, and in addition, give some others which are interesting as they bring out similarities and differences between the partial ordering of real matrices and the total ordering of real numbers.

The proof of relations 1 - 5 is obvious.

4.2. Let A_1, A_2, B_1, B_2, be matrices of similar shape.

1. If $A_1 \geqq B_1$, $A_2 \geqq B_2$,

then $A_1 + A_2 \geqq B_1 + B_2$.

2. If $A_1 > B_1$, $A_2 \geqq B_2$,

then $A_1 + A_2 > B_1 + B_2$.

3. If $A_1 \succ B_1$, $A_2 \geqq B_2$,

then $A_1 + A_2 \succ B_1 + B_2$.

4. If $A \geqq B \geqq 0$ and $\alpha \geq \beta \geq 0$,

then $\alpha A \geq \beta B$.

Let A, B, be conformable matrices (the product $A\,B$ exists).

5. If $A \geq 0$, $B \geq 0$,

then $AB \geq 0$.

<u>Note</u>. If $A \supset 0$, $B \supset 0$

then it is possible that $AB = 0$.

6. If $A \supset 0$, $B > 0$,

then $AB \supset 0$.

There is an $a_{ij} > 0$, say $a_{hk} > 0$.

Let A be $n \times m$ and let $C = AB$.

Then $c_{hl} = \sum_{j=1}^{m} a_{hj} b_{jl} > 0$.

By 5. $C = AB \geq 0$,

and hence $AB \supset 0$.

7. If $A \geq 0$, $B \geq 0$, and $AB = 0$,

then either $A = 0$, or $B = 0$.

For by 6: If $A \supset 0$, $B > 0$, then $AB \supset 0$.

Let A and B be square matrices of order n.

8. If $A \supset 0$, $B \supset 0$, and B is irreducible,

then $AB \supset 0$.

Every row of an irreducible matrix contains a non-zero non-diagonal element.

For if $b_{rj} = 0$, $j = 1, \ldots, n$, $j \neq r$,

then $b_{ij} = 0$ when $j \in G$, $i \in E - G$,

where E is the set $(1, 2, \ldots, n)$ and $j \in G$, when $j \neq r$, $j \in E$.

Hence B is reducible by the definition of 3.1.

There is some $a_{ij} > 0$, say $a_{hk} > 0$.

Suppose $b_{kl} > 0$, and let $C = A B$.

Then $c_{hl} = \sum_{j=1}^{n} a_{hj} b_{jl} \geq a_{hk} b_{kl} > 0$.

By 5, $C = A B \geq 0$,

and so $A B > 0$.

Similarly, if $A \supset 0$, $B \supset 0$ and A is irreducible, then $A B \supset 0$.

It is possible that both $A \supset 0$ and $B \supset 0$ are irreducible and yet $A B$ is not irreducible,

e.g. $A = B = \begin{bmatrix} \cdot & 1 \\ 1 & \cdot \end{bmatrix}$, $A B = \begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix}$.

9. If $A \supset 0$, $B \supset 0$, and B is non-singular, then $A B \supset 0$.

It is sufficient to prove that every row of B contains a

#h non-zero (and hence positive) element. The result then follows as in 8.

If B is non- singular, it has an inverse, P , say. We have $B P = I$.

Hence $\sum_{j=1}^{n} b_{ij} p_{ji} = 1$,

and hence there is a j , $1 \leq j \leq n$, such that $b_{ij} \neq 0$.

Similarly, if $A \supset 0$, $B \supset 0$ and A is irreducible, then $A B \supset 0$.

10. If $A > 0$, $B \supset 0$, and B is irreducible, then $A B > 0$.

When B is irreducible, any column of B contains a non-zero, non-diagonal element. The proof is similar to that for the corresponding result for rows of B.

Suppose $b_{h_j j} > 0$, $j = 1, \ldots, n$,

~~Suppose~~ and let $C = A\,B$.

C/ Then $c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj} \geqq a_{i h_j}\, b_{h_j j} > 0$,

and the result follows.

11. If $A > 0$, $B \supset 0$, and B is non-singular, then $A\,B > 0$.

Any column of a non-singular matrix contains a non-zero element. The result follows as in 10.

<u>Note</u>. The inequalities, 5 to 11, give rise to others when A is replaced by $A_1 - A_2$ or B by $B_1 - B_2$; e.g. we obtain from 5 :

If $A \geqq 0$, and $B_1 \geqq B_2$,

then $A\,B_1 \geqq A\,B_2$.

4.3. Now let A be a matrix with complex elements.

Let A^* be the matrix $A : \left[\,|a_{ij}|\,\right]$. Then $A^* \supseteq 0$, and we shall call A^* "the modulus matrix of A".

Let A, B be of similar shape.

12. $(A + B)^* \leqq A^* + B^*$.

For if $C = (A + B)^*$, $G = (A^* + B^*)$,

then $c_{ij} = |a_{ij} + b_{ij}| \le |a_{ij}| + |b_{ij}| = g_{ij}$,
$i, j = 1, \ldots, n$.

13. $(A - B)^{\times} \supseteq (A^{\times} - B^{\times})^{\times}$

Proof similar to 12.

14. Let A , B , (and hence $A^{\times}$, $B^{\times}$) be conformal.

Then $(AB)^{\times} \le A^{\times} B^{\times}$.

For let $C = (AB)^{\times}$, $G = A^{\times} B^{\times}$;

then $c_{ij} = \left|\sum_{k=1}^{n} a_{ik} b_{kj}\right| \le \sum_{k=1}^{n} |a_{ik}|\, |b_{kj}| = g_{ij}$.

15. Let D , T , be diagonal matrices.

Then $(D A T)^{\times} = D^{\times} A^{\times} T^{\times}$.

Let $D = \text{diag.}\,[d_1, \ldots, d_n]$, $T = \text{diag.}\,[t_1, \ldots, t_n]$.

We have

$$|d_i \cdot a_{ij} \cdot t_j| = |d_i|\, |a_{ij}|\, |t_j| ,$$

and the result follows.

16. Let D_1 , D_2 , D_3 , be diagonal matrices, whose diagonal elements are of unit modulus.

Let $A = D_1 M D_2$ $B = D_2^{-1} N D_3$,

where M , N , are positive matrices.

Then $(AB)^{\times} = (D_1 A B D_3)^{\times} = A^{\times} B^{\times} = MN$.

Let $D_i = \text{diag.}\,[d_1^{e_i}, d_2^{e_i} \ldots d_n^{e_i}]$, $i = 1, 2, 3$;

then $|a_{ij}| = |d_i^{e_1} m_{ij} d_j^{e_2}| = m_{ij}$.

Hence $A^{\times} = M$.

Similarly $B^{\times} = N$,

and $\quad (A\,B)^{*} = M\,N$.

4.4.

17. Let $A > 0$ be a square $n \times n$ matrix.

Let $\quad a_r \geq b_r \geq 0 \,, \quad r = 0, 1, 2, \ldots\ldots\ldots$.

Then $$\sum_{r=0}^{m} a_r A^r \geq \sum_{r=0}^{m} b_r A^r \geq 0 \,,$$

$$\sum_{r=0}^{\infty} a_r A^r \geq \sum_{r=0}^{\infty} b_r A^r \geq 0 \,,$$

and $\sum_{r=0}^{\infty} b_r A^r$ converges if $\sum_{r=0}^{\infty} a_r A^r$ converges,

$\sum_{r=0}^{\infty} a_r A^r$ diverges if $\sum_{r=0}^{\infty} b_r A^r$ diverges .

As $A^r > 0$,

$$a_r A^r \geq b_r A^r \,, \qquad \text{by } 4;$$

and the first result follows by repeated application of 1. We let $m \to \infty$, and obtain the second result by the comparison test applied to corresponding elements of

$$\sum_{r=0}^{\infty} a_r A^r \,,\ \sum_{r=0}^{\infty} b_r A^r .$$

18. If $A \geq B^{*}$, B being a matrix with complex elements,

then $$\sum_{r=0}^{\infty} a_r A^r \geq \left(\sum_{r=0}^{\infty} b_r B^r \right)^{*} ,$$

and $\sum_{r=0}^{\infty} b_r B^r$ converges if $\sum_{r=0}^{\infty} a_r A^r$ converges,

$\sum_{r=0}^{\infty} a_r A^r$ diverges if $\sum_{r=0}^{\infty} b_r B^r$ does not converge.

For $$\sum_{r=0}^{m} a_r A^r \geq \sum_{r=0}^{m} b_r B^{*r} \geq \left(\sum_{r=0}^{m} b_r B^r \right)^{*} ,$$

by 12, 4 and 1.

The rest of the proof is similar to that of 17.

4.5. We shall give ~~now~~ some examples of these results. The matrices A , B will be assumed to be square matrices of order n . The symbol Σ will be used for summation from 0 to ∞.

Let $e^X = \Sigma(X^r/r!)$

Then e^X converges for all X (c.f. Turnbull and Aitken (1932) chapter XI, and c.f. 6.10.)

19. If $A \supseteq B^{\times}$,

then $e^A \supseteq (e^B)^{\times}$.

For if $a_r = b_r = 1/r!$, then

$$e^A = \Sigma\, a_r A^r \quad , \quad e^B = \Sigma\, b_r B^r ,$$

and the result follows by 18.

If all the latent roots of A , λ_i , $i = 1 \ldots n$ satisfy $|\lambda_i| < 1$, then

$$(I - A)^{-p} = I + pA + \{p(p+1)/2!\}A^2 + \ldots ,$$

where p is any integer.

(For explanation of latent root see chapter 6, and for the proof of this result when $p \geq 0$, see 6.12.)

Let $|\mu_i| < 1$, $i = 1, \ldots, n$, and define

$$(I - A)^{-p} = I + p A + \{ p(p+1)/2! \} A^2 + \dots + \dots$$

for all $p \geq 0$.

The series converges ; c.f. 6.10.

20. If $A > 0$, and all latent roots λ_i of A satisfy $|\lambda_i| < 1$,

then $(I - A)^{-p} > (I - A)^{-q}$ when $p > q \geq 0$.

For $(I - A)^{-p} = I + p A + \{ p(p+1)/2! \} A^2 + \dots$,

$(I - A)^{-q} = I + q A + \{ q(q+1)/2! \} A^2 + \dots$,

and the result follows by 17 and $p A > q A$.

~~We have~~

21. Let $A \supset 0$ and let the latent roots of A, λ_i, satisfy

$$|\lambda_i| < 1 \quad , \quad i = 1, \dots, n.$$

If $A \supseteq B$,

then

$$(I - A)^{-p} \supseteq ((I - B)^{-p})^* \supseteq ((I - B)^{-q})^*,$$
$$p \geq q \geq 0 .$$

For $(I + p A + \{p (p + 1)/2!\} A^2 + \dots)$

$\supseteq (I + p B^* + \{p(p + 1)/2!\} B^{*2} + \dots + \dots)$

$\supseteq (I + q B^* + \{q(q + 1)/2!\} B^{*2} + \dots + \dots)^*$

$\supseteq (I + q B + \{q(q + 1)/2!\} B^2 + \dots + \dots)^*$, Note. The implication is that the series for $(I - B)^{-p}$ converges. It is easily proved from this that the moduli of latent roots of B do not exceed the greatest of the moduli of latent roots of A. But we cannot enter into this here.

a/

22. Let $A \geq 0$, $|\lambda_i| < 1$, for all latent roots of A:

then

$$(I - A)^{-1} \geq I - \log(I - A) \geq e^A \geq I + A \geq I > 0,$$

where $\log(I - A)$ is defined as

$$\log(I - A) = -\sum_{r=1}^{\infty} (A^r / r),$$

a series which converges when $|\lambda_i| < 1$, $i = 1, \ldots, n$.

$$(I - A)^{-1} = I + A + A^2 + A^3 + \cdots,$$

$$I - \log(I - A) = I + A/1 + A^2/2 + A^3/3 + \cdots,$$

$$e^A = I + A/1! + A^2/2! + A^3/3! + \cdots,$$

$$I + A = I + A.$$

The result follows by 17.

It is interesting to ~~examine~~ investigate when the equalities hold in some of the results of 22.

23. Under the conditions of 22,

(i) $(I - A) \geq I - \log(I - A)$,

(ii) $I - \log(I - A) \geq e^A$,

(iii) $e^A \geq I + A$.

The equalities hold in any one of (i), (ii), (iii), only if $A^2 = 0$.

If $A^2 = 0$, the equality holds in (i), (ii), (iii).

Consider (i) :

$$(I - A)^{-1} = I + A + A^2 + A^3 + \dots + \cdots,$$

$$I - \log (I - A) = I + A/1 + A^2/2 + A^3/3 + \dots + \cdots.$$

If $A^2 \supset 0$, $A^2 \supset A^2/2$,

and as $A^r \supseteq A^r/r$

we can not have an equality.

If $A^2 = 0$ then $A^r = 0$, $r > 2$,

and the equality holds.

The result for (ii) and (iii) is proved similarly.

4.6. The inequalities of 22 are the analogues of

$$(1 - x)^{-1} > e^x, \quad 0 \leq x < 1,$$

and $e^x \geq 1 + x$, for all real x.

It is natural to enquire whether we also obtain analogues to

$$(1 - x) \leq e^{-x} \leq (1 + x)^{-1}, \quad 0 \leq x < 1.$$

Suppose that the latent roots of A satisfy $|\lambda_i| < 1$, $i = 1, \dots, n$,

and that $A \supseteq 0$.

It is clear that

$$(I - A) \subseteq e^{-A} \subseteq (I + A)^{-1}$$

does <u>not</u> hold in general.

For example, Let $A = V$, the 4×4 auxiliary unit matrix,

$$V = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}.$$

Then

$$e^{-V} = (I - V + V^2/2! - V^3/3!),$$

as $V^r = 0$, $r \geqslant 4$.

(c.f. chapter 6, 10).

Thus

$$e^{-V} = \begin{bmatrix} 1 & -1 & \frac{1}{2} & -\frac{1}{6} \\ & 1 & -1 & \frac{1}{2} \\ & & 1 & -1 \\ & & & 1 \end{bmatrix},$$

while

$$I - V = \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix},$$

and hence $e^{-V} \neq I - V$.

Similarly $(I + V)^{-1} = I - V + V^2 - V^3$.

Thus

$$(I + V)^{-1} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ & 1 & -1 & 1 \\ & & 1 & -1 \\ & & & 1 \end{bmatrix},$$

and so $\quad (I - V) \parallel e^{-V}$,

$$(I + V)^{-1} \parallel I - V .$$

4.7. However, there are "negative" analogues of some of the properties of non-negative numbers.

24. Let $A \geqslant 0$, and let $(I + A)$ be non-singular.

Then $\quad (I - A) \not\supset (I + A)^{-1}$.

Suppose $(I - A) \supset (I + A)^{-1}$.

Then $(I - A)(I + A) \supset I$, from 9, c.f. note at the end of 4.2.

Hence $\quad I - A^2 \supset I$,

and this is impossible.

Hence $\quad (I - A) \not\supset (I + A)^{-1}$.

25. Let $A \supset 0$.

If $A^2 = 0$; then $(I - A) = (I + A)^{-1}$.

If $A^2 \supset 0$, then $(I - A) \not\supset (I + A)^{-1}$.

The first statement follows from

$$(I - A)(I + A) = I - A^2 = I \text{ when } A^2 = 0.$$

The proof of the second is similar to that of 24. We use 5, in place of 9.

26. Let $A \supset 0$.

Then $(I - A) \not\supset e^{-A}$

Suppose $\quad (I - A) \supset e^{-A}$.

We have $\quad e^{-A} > 0$,

Hence

$$I = e^{-A} e^{+A} \subset (I - A) e^{A} = I + (A - A) + (A^2/2 - A)$$
$$+ (A^3/3 - A^3/2) + \cdots \leq I ,$$

by 9, and the absolute convergence of the series.

This is not possible.

The result follows.

27. Let $A \supset 0$.

If $A^2 = 0$, then $I - A = e^{-A}$.

If $A^2 \supset 0$, then $(I - A) \not\supset e^{-A}$.

Proofs similar to 25.

CHAPTER 5.

5.1. In Chapter 3 we considered chains whose members were submatrices. In this chapter we shall consider chains whose members are elements of a matrix A. We shall be chiefly concerned with the "length" of chains.

Definition : A chain is defined as a sequence $a_{i_1 i_2}$, $a_{i_2 i_3}$, , $a_{i_{s-1} i_s}$, of non-zero elements of A. We shall denote this sequence by $(i_1\ i_2,\ ,i_s)$ where a comma is inserted when some element is not explicitly written. Thus (i , j) denotes a chain beginning with i and ending with j. We note that we have not excluded diagonal elements a_{ii} from being members of a chain. This is a distinction between the present case and that of chapter 3. A chain (i , i) will be called a cycle.

The length of a chain is defined as the integer equal to the number of members of the chain, minus one.

If $a_{ij} \neq 0$, we shall put $\ell_{ij} = 1$,

If $a_{ij} = 0$, we shall put $\ell_{ij} = 0$.

Let $L_{ij}^{(s-1)} = \sum \ell_{i_1 i_2}\ell_{i_2 i_3} \ldots \ell_{i_{s-1} i_s}$, $i_1 = i$, $i_s = j$,

summed over all distinct sequences $i_1, i_2, \ldots, i_s$, for fixed $s \geq 2$.

Let $$L_{ij} = \sum_{s=1}^{\infty} L_{ij}^{(s)} .$$

We may note that we shall often have $L_{ij}^{(s)}$, L_{ij} , infinite. This is of no importance; c.f. 3.3 .

Theorem 1. There is an (i, j) of length s if and only if $L^{(s)}_{ij} > 0$.

If $L^{(s)}_{ij} > 0$, then there is a term $l_{i_1 i_2} \cdots\cdots l_{i_s i_{s+1}} > 0$. where $i_1 = i$, $i_{s+1} = j$.

Thus $l_{i_1 i_2} = 1$, $l_{i_2 i_3} = 1$, . . . , $l_{i_s i_{s+1}} = 1$;

or $a_{i_1 i_2} \neq 0$, $a_{i_2 i_3} \neq 0$, . . . , $a_{i_s i_{s+1}} \neq 0$,

whence, by definition, $(i_1, i_2, \quad , i_{s+1})$ is a chain.

We reverse the argument and obtain the converse.

Corollary. There is an (i, j) if and only if $L_{ij} > 0$.

We have $L_{ij} > 0$ if and only if some $L^{(s)}_{ij} > 0$.

There is an (i, j) if and only if there is an s such that there is an (i, j) of length s.

The corollary now follows from Theorem 1.

It will be convenient to assume that every irreducible matrix is non-zero. An irreducible matrix is zero if and only if it is the 1×1 null-matrix. Should such a matrix occur in the decomposition of A we shall agree to replace it by a non-zero 1×1 matrix. It is easily seen, that without this, theorem 2, etc. would not hold for a null 1×1 matrix.

Suppose there is a chain (i, j), $(i_1, i_2, \ldots, i_s)$ say, where $i_1 = i$, $i_s = j$.

If $i_r = i_t$, we may omit $i_{r+1}, i_{r+2}, \ldots i_t$, and still have an (i, j). Repeating this process we obtain an (i, j),

$(i, j) = (i_1' \; i_2', \ldots\ldots i_v')$ say, with distinct i_w', except perhaps that $i_*^* = i_1' = i_v' = j$.

5.2.

Lemma. <u>If $L_{ii} = 0$, either $L_{ij} = 0$, or $L_{ji} = 0$, $j = 1, \ldots n$.</u>

For $L_{ii} = L_{ij} L_{ji}$ + (non-negative terms).
Hence if $L_{ii} = 0$,
then $L_{ij} L_{ji} = 0$,
and the lemma follows.

<u>Theorem 2.</u> <u>The matrix A is irreducible if and only if either</u>

<u>(1) $L_{ij} > 0$, $i,j = 1, 2, \ldots, n$,</u> <u>or</u>

<u>(2) There is an i such that $L_{ij} > 0$, for $j = 1, 2, \ldots n$.</u>

Clearly (1) implies (2). Thus we need only prove that (1) is necessary, and that (2) is sufficient. We may restate these:

$(1)'$ If there are i,j, $1 \leq i,j \leq n$, such that $L_{ij} = 0$, then A is reducible.

$(2)'$ If A is reducible, then given any i, $1 \leq i \leq n$, there is a j, $1 \leq j \leq n$, such that $L_{ij} = 0$.

<u>Proof of $(1)'$:</u> Let $L_{ij} = 0$.
If $i = j$ there is a $k \neq i$ such that either $L_{ik} = 0$ or $L_{ki} = 0$, by the lemma. We may therefore suppose that

$L_{ij} = 0$ and $i \neq j$.

Let E be the set (1, 2, ..., n), and let G be the subset of E such that $i \varepsilon G$, and $k \varepsilon G$ when $L_{ik} > 0$, and $k \neq i$. Then G is a proper subset of E as $i \varepsilon G$ and $j \varepsilon E - G$.

Let $l \varepsilon E - G$.

$L_{il} = l_{il}$ + non-negative terms = 0.

Hence $l_{il} = 0$.

Let $k \varepsilon G$ and $k \neq i$, while $l \varepsilon E - G$.

$$L_{il} = L_{ik} \, l_{kl} + \text{(non-negative terms)} = 0.$$

Hence $l_{kl} = 0$ as $L_{ik} > 0$.

Thus $a_{kl} = 0$, when $k \varepsilon G$, $l \varepsilon E - G$, whence A is reducible.

Proof of (2)':

Let A be reducible. There is a proper subset of E, say G, such that $a_{kj} = 0$, $k \varepsilon G$, $j \varepsilon E - G$.

Suppose $i \varepsilon G$, $j \varepsilon E - G$. Then $l_{ij} = 0$, and each term of L_{ij} contains a zero factor, as in $l_{i_1 i_2} \cdots l_{i_{s-1} i_s}$, $i_1 = i$, $i_s = j$, there must be an k for which $i_k \varepsilon G$ $i_{k+1} \varepsilon E - G$.

Hence $L_{ij} = 0$.

Corollary:

(1) If A is irreducible then there is a cycle (i, i) for any $i \varepsilon E$.

(2) If A is irreducible then $L_{ii} > 0$ for any $i \varepsilon E$.

By theorem 2, $L_{ij} > 0$, for any $i, j \in E$ if A is irreducible.

Putting $j = i$ we obtain (2), and (2) is equivalent to (1), by theorem 1, Corollary.

5.3.

Let E_α, $R_{\alpha\beta}$, $r_{\alpha\beta}$ be defined as in chapter 3.

<u>Theorem 3.</u> <u>Let $i \in E_\alpha$, $j \in E_\beta$; then $L_{ij} > 0$ if and only if $R_{\alpha\beta} > 0$.</u>

This theorem states (by 3.8 and 5, Theorem 1) that there is a chain (i , j) of elements if and only if a_{ii}, a_{jj} lie in irreducible submatrices between which there is a chain of submatrices.

Suppose $L_{ij} > 0$. If $\alpha = \beta$, $R_{\alpha\beta} > 0$ by definition.

Let $\alpha \neq \beta$.

There is a non-zero term $l_{i_1 i_2} l_{i_2 i_3} \ldots l_{i_{s-1} i_s}, i_1 = i, i_s = j$.

Now suppose $i_k \in E_{\alpha_k}$.

Let $(\gamma_1, \ldots, \gamma_t)$ be obtained from this sequence $(\alpha_1, \ldots, \alpha_s)$ by omitting all repetitions, (i.e. $\gamma_r \neq \gamma_{r+1}$), and let $(h_1, \ldots, h_t)$ be obtained from $(i_1, \ldots, i_s)$ by omitting the corresponding terms.

Then $a_{h_p h_{p+1}} \neq 0$ as $l_{h_p h_{p+1}} = 1$,

Hence $A_{\gamma_p \gamma_{p+1}} \neq 0$,

where $A_{\gamma_p \gamma_{p+1}} : [a_{ij}],\ i \in E_{\gamma_p},\ j \in E_{\gamma_{p+1}}$.

Hence $r_{\gamma_p \gamma_{p+1}} = 1$,

and $R_{\alpha\beta} = R_{\gamma_1\gamma_t} = r_{\gamma_1\gamma_2} \cdots r_{\gamma_{t-1}\gamma_t}$ + (non-negative terms) ≥ 0.

Suppose $R_{\alpha\beta} > 0$,

If $\alpha = \beta$, the result follows by theorem 2 (1).

Let $\alpha \neq \beta$.

There are distinct $\gamma_1 \ldots\ldots \gamma_t$, $\gamma_1 = \alpha$, $\gamma_t = \beta$ such that

$$r_{\gamma_1\gamma_2} r_{\gamma_2\gamma_3} \cdots r_{\gamma_{t-1}\gamma_t} > 0.$$

It follows that

$A_{\gamma_p\gamma_{p+1}} \neq 0$, $(1 \leq p < t)$, whence there is a $j_p \in E_p$, $i_{p+1} \in E_{p+1}$ such that

$a_{j_p i_{p+1}} \neq 0$. This is equivalent to $L_{j_p i_{p+1}} \neq 0$.

But $A_{\gamma_p\gamma_p}$ is irreducible. Hence if $i_p, j_p \in E_{\gamma_p}$, then $L_{i_p j_p} > 0$ by Theorem 2.

Putting $i_1 = i$, $j_t = j$ we obtain

$$L_{ij} \geq L_{i_1 j_1} L_{j_1 i_2} L_{i_2 j_2} \cdots L_{i_t j_t} > 0.$$

<u>Theorem 4.</u> <u>There is a cycle (i, j, i) if and only if there is an α such that $i, j \in E_\alpha$.</u>

This theorem states that all elements of a cycle in A lie within an irreducible $A_{\alpha\alpha}$.

<u>Proof.</u>

Let there be a cycle (i, j, i).

$L_{ij} > 0$, $L_{ji} > 0$ by theorem 1, Corollary.

Let $i \in E_\alpha$, $j \in E_\beta$.

By theorem 3 $R_{\alpha\beta} > 0$, $R_{\beta\alpha} > 0$.

Hence by 3, Theorem 2, $\alpha \geq \beta$, $\beta \geq \alpha$, whence $\alpha = \beta$.

By theorem 2, the converse is true, as $L_{ij} > 0$, $L_{ji} > 0$ when $i, j \in E_\alpha$.

5.4.

We shall define $a_{ij}^{(p)}$ as $a_{ij}^{(p)} = b_{ij}$, when $B = A^p$.

<u>Theorem 5.</u> <u>If $A \supset 0$,</u>

<u>then $a_{ij}^{(p)} > 0$ if and only if ~~there is~~ $L_{ij}^{(p)} > 0$.</u>

(i) If $p = 1$, this is true by definition.

Suppose the theorem is true for $p = r \geqslant 1$.

We have $a_{ij}^{(r+1)} = \sum_{\mu=1}^{n} a_{i\mu}^{(r)} a_{\mu j}$:

(ii) Whence $a_{ij}^{(r+1)} > 0$ if and only if there is a μ such that $a_{i\mu}^{(r)} > 0$ $a_{\mu j} > 0$,

i.e. if and only if there is a μ such that

$$L_{i\mu}^{(r)} > 0 \qquad L_{\mu j} > 0 .$$

Hence $L_{ij}^{(r+1)} = \sum_{\mu=1}^{n} L_{i\mu}^{(r)} L_{\mu j} > 0$

if and only if $a_{ij}^{(r+1)} > 0$.

The theorem follows by induction.

If the condition $A \supset 0$ is dropped, we obtain theorem 5 a.

<u>Theorem 5 a.</u> <u>If $a_{ij}^{(p)} \neq 0$, $L_{ij}^{(p)} > 0$.</u>

We repeat (i), and in (ii) we must replace "if and only if" by "only if" and

$$\text{"}a_{ij}^{(s} > 0\text{"} \text{ by } \text{"}a_{ij}^{(s} \neq 0,\text{"} \quad s = 1, \ldots, r, r+1 \ .$$

5.5.

A is an irreducible $n \times n$ matrix. A will be said to be "of modulus k" (mod $A = k$) when k is the largest number of sets satisfying conditions 3.2, (a), (b), (c), and

5.5 (a) : $A_{\alpha\beta} = 0$ if $\beta \not\equiv \alpha + 1$, mod k,

where $A_{\alpha\beta} : [a_{ij}]$, $i \in E_\alpha$, $j \in E_\beta$, $E_\alpha = E_{\alpha'}$, when $\alpha' \equiv \alpha$ mod k, and $j \equiv i$ mod k, when there is an integer r such that $j - i = rk$.

When $E_1 = E$, and $k = 1$, E satisfies 3.2. (a), (b), ((c) does not apply) and also 5.5 (a) as

$$A_{12} : [a_{ij}], \ i \in E_1 = E, \quad j \in E_2 = E_1 = E,$$

so that $A_{12} = A$.

Hence mod $A \geq 1$.

When mod $A = 1$, A is called "primitive", otherwise A is called "imprimitive".

Theorem 6. The modulus of an irreducible A equals the greatest common divisor of the lengths of all cycles in A.

(a) By theorem 1, p, the g.c.d. of all cycles in A, is the g.c.d. of integers s for which some $L_{ii}^{(s}$ is non-zero.

Let mod $A = k$, the sets $E_1, E_2, \ldots, E_k$, being defined by 3.2, (a), (b), (c), and 5.5. (a).

Then $a_{ij} \neq 0$ only if $i \in E_\alpha$, $j \in E_{\alpha+1}$, where $\alpha + 1$

is to be interpreted as $\alpha + 1$, mod k .

If for some i, $L^{(s)}_{ii} > 0$, suppose there is a cycle

$(i_1\ i_2, \ldots, i_s\ i_1)$, $i_1 = i$. Hence if $i_1 \in E_\alpha$, then $i_2 \in E_{\alpha+1}$, ~~then~~ ...

$i_s \in E_{\alpha+s-1}$ and $i \in E_{\alpha+s} = E_\alpha$.

Whence $s \equiv 0$, mod k.

All s , for which some $L^{(s)}_{ii} > 0$, are divisible by k . Hence $p = rk$, where r is an integer, $r \geq 1$.

(b) Now suppose $L^{(s)}_{ij} > 0$, $L^{(t)}_{ij} > 0$. We shall prove that

$$s \equiv t, \bmod p .$$

As A is irreducible there is an (j, i) of length u , say. Hence $L^{(u)}_{ji} > 0$.

$$L^{(s+u)}_{ii} = L^{(s)}_{ij} L^{(u)}_{ji} + (\text{non-negative terms}) > 0 .$$

Similarly $L^{(t+u)}_{ii} > 0$.

Hence $s + u \equiv t + u$, mod p ,

and the result follows.

(c) ~~Now~~ Let $i \in E_1$, $(1 \leq t \leq p)$, and if $L^{(s)}_{ij} > 0$, let $j \in E_t$ ~~Let s be ... where $t \leq p$~~, and $t \equiv 1 + s$, mod p

As there is a chain (i, j) for all j , we must have

$$E_1 \cup E_2 \cup \ldots \cup E_p = E.$$

As there is some j for which $L^{(\alpha-1)}_{ij} > 0$, $1 \leq \alpha \leq p$, hence $E_\alpha \neq 0$.

Let $j \in E_\alpha$, and $\beta \neq \alpha$, $1 \leq \beta, \alpha \leq p$.

There is an s such that $L^{(s)}_{ij} = 0$, $s \equiv \alpha - 1$, mod p.

Hence by (b) $L^{(t)}_{ij} = 0$ when $t \equiv \beta - 1$ mod p,

and hence $j \notin E_\beta$.

Hence $E_\alpha \cap E_\beta = 0$, $1 \leq \alpha, \beta \leq p.$, $\beta \neq \alpha$.

It follows that 3.2, (a), (b), (c) are satisfied.

(d) Suppose $i \in E_\alpha$, and $a_{ij} \neq 0$.

Then $L^{ij}_{ij} = l_{ij} > 0$, and hence, by (c),

$j \in E_{\alpha+1}$, where $\alpha+1$ may be interpreted as $\alpha + 1$ (mod p).

It follows that $A_{\alpha\beta} = 0$, if $\beta \not\equiv \alpha + 1$, mod p,
where $A_{\alpha\beta} = [a_{ij}]$, $i \in E_\alpha$, $j \in E_\beta$.

Hence 5.5. (a) is satisfied by the sets $E_1, E_2, \ldots. E_p$, constructed in (c).

Hence $k \geq p$.

But by (a) $p \geq k$, whence $p = k$.

<u>Corollary.</u>

<u>If $i \in E_\alpha$, then $j \in E_\beta$, if and only if there is an $s \equiv \beta-\alpha$, mod k, where $L^{is}_{ij} > 0$.</u>

We have essentially used this corollary in (c) above.

Let $L^{is}_{ij} > 0$, and $s \equiv \beta - \alpha$ mod k.

Suppose $s = \beta - \alpha + rk$.

There is a chain $(i_1, \ldots, i_{\beta-\alpha+rk+1})$; $i_1 = i$,
$i_{\beta-\alpha+rk+1} = j$.

As in (a), $j = i_{\beta-\alpha+rk+1} \in E_{\beta+rk} = E_\beta$.

Conversely,
Let $i \in E_\alpha$, and $j \in E_\beta$.

As A is irreducible, there is a chain (i, j).

Let this chain be of length s. Then $L^{is}_{ij} > 0$.

As before $j \in E_{s+\alpha}$.

Hence $s = \beta - \alpha$, mod k.

5.6. Theorem 7.

Let $F_1, F_2, \ldots, F_q$ satisfy 3.2 (a), (b), (c) and 5.5. (a).

Then (1) q divides k, k = mod A,

$$(2) \quad F_\alpha = \sum_{r=0}^{h-1} E_{\alpha+\beta+rq},$$

where $hq = k$, and β does not depend on α.

(a) If q divides k we shall write (q / k).
The argument is exactly that of theorem 6 (a), with q written for k.
By theorem 6, $p = k$ and we obtain $k = hq$, for some h.

(b) Suppose $L^{(s)}_{ij} > 0$, $L^{(t)}_{ij} > 0$.
By theorem 6, (b),

$$s \equiv t, \quad \text{mod } k,$$

whence by (a) $s \equiv t$ mod q.

We may now prove, exactly as we proved the corollary of theorem 6, that if $i \in F_\alpha$, then, $j \in F_\beta$ if and only if there is such an s such that

$$L^{(s)}_{ij} > 0 \quad \text{and} \quad s \equiv \beta - \alpha, \text{ mod } q.$$

Now suppose $i \in F_1$, $j \in F_\alpha$, and $i \in E_{\beta+1}$,
where $E_1, E_2, \ldots E_k$ are the sets of theorem 6.
There is such an s that $L^{(s)}_{ij} > 0$.

Hence $s \equiv \alpha - 1$, mod q ,

and as $k = h\,q$,

$s \equiv \alpha - 1 + r\,q$, mod k ,

where $0 \leq r \leq h - 1$.

By theorem 6, Corollary ,

$$j \in E_{\beta+1+s} = E_{\alpha+rq+\beta} .$$

As this holds for any $j \in F_\alpha$,

$$F_\alpha \supset \sum_{r=0}^{h-1} E_{\alpha+\beta+rq} = G_\alpha , \text{ say.}$$

But $E = \sum_{\alpha=1}^{q} F_\alpha = \sum_{\gamma=1}^{*} E_\gamma = \sum_{\alpha=1}^{q} G_\alpha$,

Hence $F_\alpha = G_\alpha$,

and this proves the theorem .

Corollary.

If the sets $E_1, E_2, \ldots, E_k$, $F_1, F_2, \ldots, F_k$, satisfy 3.2, (a), (b), (c), and $k =$ mod A , there is a β such that $F_\alpha = E_{\alpha+\beta}$ $\alpha = 1, \ldots, k$.

Putting $k = q$ we immediately obtain the corollary from theorem 7.

In other words, the sets $E_1, \ldots E_k$ of theorem 6 are unique apart from a possible trivial renumeration.

5.7.

If A is reducible and in normal form,

$$A = \begin{bmatrix} A_{11} & & & \\ A_{21} & A_{22} & & \\ \vdots & & \ddots & \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{bmatrix},$$

then $$f(A) = \begin{bmatrix} f(A_{11}) & & & \\ F_{21} & f(A_{22}) & & \\ \vdots & & \ddots & \\ F_{r1} & F_{r2} & \cdots & f(A_{rr}) \end{bmatrix},$$

where $f(A)$ is any scalar polynomial in A, and the result is obtained by simply multiplying out $f(A)$. The $f(A_{ii})$ are not necessarily irreducible. That the normal form of $f(A)$ is as given above may also be shown thus:

Let $i \in E_\alpha$, $j \in E_\beta$.

If $\alpha < \beta$, $R_{\alpha\beta} = 0$, hence by theorem 3 $L_{ij} = 0$.

As $L_{ij} = L^{i1}_{ij} + L^{i2}_{ij} + L^{i3}_{ij} + + = 0$,

$L^{ip}_{ij} = 0$, $p = 1, 2, \ldots$.

Hence $a^{ip}_{ij} = 0$, by theorem 5.

Hence ~~f(A) ≠ 0~~ $F_{\alpha\beta} = 0$, $\alpha < \beta$.

Now suppose A is irreducible and mod $A > 1$.

By a conjugate permutation of rows and columns (i.e. putting first the $i \in E_1$, equal to 1, 2, ..., then the $i \in E_2$ etc. where the E_α are the sets of 5.5), we have

$$A = \begin{bmatrix} \cdot & A_{12} & & & \\ & \cdot & A_{23} & & \\ & & \cdot & \ddots & \\ & & & \cdot & A_{k-1\,k} \\ A_{k1} & & & & \cdot \end{bmatrix},$$

where we have indicated only non-zero submatrices.

As $a_{ij}^{(p)} \neq 0$, only if $L_{ij}^{(p)} > 0$, and hence only if $\beta - \alpha \equiv p, \bmod k$, $i \in E_\alpha$, $j \in E_\beta$, it follows that ~~Then~~ if $p \equiv q, \bmod k$, $1 \leq q \leq k$,

$$A^{p} = \begin{bmatrix} \cdot & \cdot & A_{1\,q+1}^{(p)} & & \\ & & \cdot & A_{2\,q+2}^{(p)} & \\ & & \cdot & & \ddots \\ & & \cdot & & A_{k-q\,k}^{(p)} \\ A_{k+1-q\,1}^{(p)} & & \cdot & & \\ & \ddots & \cdot & & \\ & & A_{k\,q}^{(p)} \quad \cdot & & \end{bmatrix}.$$

5.8.

Lemma. 2. Let $l_1, l_2, \ldots\ldots l_s$, be distinct positive integers whose g.c.d. is 1. Then there are non-negative integers x_i, $i = 1, \ldots\ldots s$, such that $\sum_{i=1}^{s} x_i l_i = t$, when $t \geq (s-1)R - \sum_{i=1}^{s} l_i + 1$ where R is the least common multiple (l.c.m.) of $l_1, l_2, \ldots\ldots l_s$.

It is well-known (cf. Birkhoff and MacLane (1941) p. 20) that there are integers x'_i, $i = 1, \ldots, s$ (positive, zero, or negative) such that

$$\sum_{i=1}^{s} x'_i l_i = t .$$

Let us suppose that $l_1 > l_2 > \quad > l_s$,

and that $l_i L_i = R$, $i = 1, \ldots s$.

There are integers μ_i such that

$$x'_i + \mu_i L_i = x_i, \quad i = 1, \ldots s-1,$$

where $0 \leq x_i \leq L_i - 1$.

Hence $t = \sum_{i=1}^{s} x_i l_i$,

where
$$x_s = x'_s - \left(\sum_{i=1}^{s-1}\mu_i\right)(R/l_s)$$
$$= x'_s - \left(\sum_{i=1}^{s-1}\mu_i\right) L_s ,$$

and $0 \leq x_i \leq L_i - 1$, $i = 1, \ldots, s-1$.

Let
$$q = \sum_{i=1}^{s-1}(L_i - 1) l_i$$
$$= (s-1)R - \sum_{i=1}^{s-1} l_i .$$

Suppose that $t \geq q$, and that $t = \sum_{i=1}^{s} x_i l_i$.

It follows from $\sum_{i=1}^{s-1} x_i l_i \leq q$ that $x_s \geq 0$.

Suppose that $q > t > q - l_s$ and that $t = \sum_{i=1}^{s} x_i l_i$.

If $x_i = L_i - 1$, $i = 1, \ldots, s-1$,

then $q > \sum_{i=1}^{s} x_i l_i = q + x_s l_s > q - l_s$,

which is impossible, for integral x_s.

Hence there is a j for which $x_j \leq L_j - 2$.

Thus $\sum_{i=1}^{s-1} x_i L_i \leq q - L_j < q - L_s < t = \sum_{i=1}^{s} x_i L_i$,

whence $x_s > 0$.

The lemma follows.

<u>Lemma 3.</u> <u>Let A be an irreducible matrix, and let $i, j, i_1, i_2, \ldots, i_s$, be integers $1 \leq i, j, i_k, \leq n$, $k = 1, \ldots, s$.</u>

<u>Then there is a chain $(i, i^*_{k_1}, i^*_{k_2}, \ldots, i^*_{k_s}, j)$ in A, where $(i_{k_1}, i_{k_2}, \ldots, i_{k_s})$ is an arrangement of $(i_1, i_2, \ldots, i_s)$, of length less or equal to $\frac{1}{2}(2n - s)(s + 1) - 1$.</u>

Let S be the set $(i_1, i_2, \ldots i_s)$, and let $h \varepsilon S$.

As A is irreducible there is a chain $(i'_0\ i'_1\ i'_2 \ldots i'_{v-1}, i'_v)$ $i'_0 = i$, $i'_v = h$, consisting of distinct members (cf. 5.1). Hence, since $i'_v \varepsilon S$, there is a t, which is the smallest integer w such that $i'_w \varepsilon S$.

As S contains s members it follows that $t \leq n - s$.

Let $i'_t = i_{k_1}$.

Thus there is a $i_{k_1} \varepsilon S$ for which there is a (i, i_{k_1}) of length less or equal to $n - s$.

Similarly there is a $i_{k_2} \varepsilon S$, $i_{k_2} \neq i_{k_1}$, such that there is a (i_{k_1}, i_{k_2}) of length less or equal to $n - s + 1$.

Continuing in this way we see that there is an arrangement $(i_{k_1}, i_{k_2}, \ldots, i_{k_s})$ of $(i_1, i_2, \ldots\ldots i_s)$ such that there is an $(i_{k_t}, i_{k_{t+1}})$, $t = 1, \ldots\ldots s - 1$, of length less or equal to $n - s + k$. Finally there is an (i_{k_s}, j) of length less or equal to $n - 1$.

It follows that there is a required chain of length less or equal to

$$\sum_{k=0}^{s}(n - s + k) - 1 = \tfrac{1}{2}(2n - s)(s + 1) - 1 .$$

This is the lemma.

5.9. Theorem 8.

Let $A > 0$. There is a positive integer p such that $A^p > 0$, if and only if A is irreducible and primitive.

If A is reducible, or irreducible and imprimitive, it follows from the normal forms of A of 5.7. that A^ρ, $\rho = 1, 2, 3 \ldots$ is not strictly positive. The second part of the theorem is implied by the last part of theorem 9.

Theorem 9.

Let $A > 0$, be irreducible and primitive. Then A^r, $(r \geq 1)$ is irreducible and primitive. There is a p such that $A^r > 0$, when $r \geq p$. (Cf. Frobenius (1912),).

The last statement will be proved first.

(a) As A is irreducible and primitive, there are cycles $\Gamma_t = (i_t, i_t)$ of length l_t, $t = 1, \dots, s$, such that the g.c.d. of $l_1, l_2, \dots l_s$, is 1, (Theorem 6).

Let R be the l.c.m. of $l_1, l_2, \dots l_s$,

and let $p_1 = (s-1)R - \sum_{t=1}^{s} l_t + 1$,

Let $p_2 = \frac{1}{2}(2n-1)(s+1) - 1$.

and let $p = p_1 + p_2$.

(b) Suppose $r \geq p$.

By lemma 3 there is an arrangement $(i_{k_1}, i_{k_2}, \dots, i_{k_s})$ of $(i_1, i_2, \dots, i_s)$ such that there is a chain $(i, i_{k_1}, i_{k_2}, \dots, i_{k_s}, j)$ of length $q \leq p_2$.

where $1 \leq i, j, \leq n$.

By lemma 2 there are non-negative integers x_t, $t = 1, \dots, s$, such that

$$\sum_{t=1}^{s} x_t l_t = r - q \geq p_1 .$$

Let $x_t \Gamma_t$ denote the cycle obtained when the sequence of integers constituting $\Gamma_t = (i_t, i_t)$ is written x_t times successively, with adjoining i_t omitted. Thus,

if $\Gamma_t = (1\ 2\ 3)$,

then $3\Gamma_t = (1\ 2\ 3\ 1\ 2\ 3\ 1\ 2\ 3\ 1)$.

The length of $x_t \Gamma_t$ is $x_t l_t$.

Let Γ be the chain $(i, i_{k_1}, x_{k_1}\Gamma_{k_1}, \dots, i_{k_s}, x_{k_s}\Gamma_{k_s}, j)$, where adjoining i_t are again omitted.

The length of Γ is $q + r - q = r$.

It follows from theorem 1 that $L_{ij}^{(r)} > 0$.

Hence, by theorem 5, $a_{ij}^{(r)} > 0$.

This result holds for all i, j $1 \leq i, j \leq n$.

Hence $A^r > 0$.

(c) We now prove that $A^r, r \geq 1$, is irreducible and primitive.

By theorem 5, $a_{ij}^{(r)} > 0$, when $L_{ij}^{(r)} > 0$.

Hence by theorem 1, $a_{ij}^{(r)} > 0$, when there is a chain (i, j) in A of length r.

There is a chain $(i_1, i_2, \ldots\ldots i_{s+1})$, $i_1 = i$, $i_{s+1} = j$, in A^r if and only if there is a sequence of $a_{i_k i_{k+1}}^{(r)} \geq 0$, $k = 1, \ldots, s$.

It follows there is a chain (i , j) in A^r, if and only if there are chains (i_k, i_{k+1}), $k = 1, \ldots\ldots s$, in A, of length r.

Hence there is a chain of length s in A^r, if and only if there is a chain (i , j) of length rs in A .

Let p be the integer defined in (a),

and let $s \geq p/r$.

Then by (b) there is an (i , j) of length rs. Hence there is an (i , j) in A of length s.

It follows that $L'_{ij} \geq L'^{(s)}_{ij} > 0$, where L'_{ij}, $L'^{(s)}_{ij}$ are defined from A^r as L_{ij}, $L^{(s)}_{ij}$ for A.

This result holds for all i, j, $1 \leq i, j \leq n$.

Hence by theorem 2 (1), A^r is irreducible.

(d) Let s_1, s_2 be two integers whose g.c.d. is 1, and let $s_k \geqslant p/r$, $k = 1, 2$.
By (c) there are cycles (i, i) in A^r of length s_1, s_2. It follows from theorem 6 that mod $A = 1$. Hence A^r is primitive.

We note that we might also have proved the first parts of theorem 8 by considering chains of reducible, or irreducible imprimitive matrices. In the proof of theorem 9 we made no attempt to find a least integer p such that $A^p > 0$.

CHAPTER 6.

The Latent Roots and the Classical Canonical Form of a Matrix.

6.1.

Let A be a square matrix of order n.

(a) A principal submatrix of A is a submatrix $M : [a_{ij}]$ $i, j \in G$, where G is a subset of $E = (1, 2, \ldots, n)$, i.e. a submatrix at the intersection of the same set of rows and columns.

The determinant of a submatrix is called a minor, that of a principal submatrix a principal minor.

(b) The solutions of $|A - \lambda I| = 0$ (1)

are called the latent roots of A.

The equation (1) is of degree n, and thus if latent roots are counted according to their multiplicities as roots of (1), A has n latent roots,

$\lambda_1, \lambda_2 \ldots\ldots \lambda_n$. Some of these may coincide.

(c) The rank of A is the order of the non-vanishing minor of A of highest order. (The determinant $|A|$ itself is considered a minor of A of order n).

(d) The non-zero vectors $x_1, x_2 \ldots\ldots x_r$, are linearly independent when the equation $\sum_{i=1}^{r} c_i x_i = 0$

implies $c_1 = c_2 = \ldots = c_r = 0$.

(e) It is well-known that the number of linearly independent non-zero solutions of

$$A x = 0 \tag{2}$$

is $\nu = n - r$, where r is the rank of A. This implies that (2) is soluble for non-zero x if and only if $|A| = 0$.

(f) It follows from (1) and (e) that there is a vector x_i such that

$$(A - \lambda_i' I) x_i = 0 \tag{3}$$

if and only if λ_i' is a latent root of A.

A vector x_i satisfying (3) is called "a latent column vector associated with λ_i'.

(g) Let $t_k(\lambda)$ be the sum of principal minors of $A - \lambda I$ of order k and $t_k(0) = t_k$, $k = 1, 2, \ldots n$.

The well-known diagonal expansion of a determinant yields

$$|A + \lambda I| = \lambda^n + t_1 \lambda^{n-1} + \qquad + t_{n-1}\lambda + t_n$$

$$= \sum_{k=0}^{n} t_{n-k} \lambda^k ,$$

where $t_0 = 1$.

Hence $|A - \lambda I| = |A - \lambda_i I + (\lambda_i - \lambda) I|$

$$= \sum_{k=0}^{n} t_{n-k} (\lambda_i) (\lambda_i - \lambda)^k ,$$

where $t_0(\lambda) = 1$.

We shall now denote by $\lambda_1, \lambda_2, \ldots \lambda_r$, the distinct latent roots of A.

(h) We deduce that λ_i is a μ-fold latent root of A, if and only if

$$t_n(\lambda_i) = t_{n-1}(\lambda_i) = \dots = t_{n-\mu+1}(\lambda_i) = 0,$$

$$t_{n-\mu}(\lambda_i) \neq 0.$$

(We note that $t_n(\lambda_i) = |A - \lambda_i I|$.)

(i) Evidently if $A - \lambda_i I$ is of rank r then $\nu = n - r \leq \mu$.

Hence $t_{r+1}(\lambda_i) = t_{r+2}(\lambda_i) \quad = t_n(\lambda_i) = 0.$

In other words the number of latent vectors associated with the latent root λ_i is smaller or equal to the multiplicity of λ_i.

(j) Non-zero solutions of $y_i'(A - \lambda_i I) = 0$

$$\text{or} \quad y_i' A = \lambda_i y_i' \qquad (4)$$

are called "latent row vectors associated with λ_i", and (4) is soluble for non-zero y_i' if and only if λ_i is a latent root of A. The number of latent row vectors associated with λ_i is also ν. These results follow by considering $(A - \lambda_i I)'$ (the transposed matrix) as

$$|A - \lambda_i I| = |(A - \lambda_i I)'|.$$

6.2.

We shall denote by $V : [v_{ij}]$, i, j = 1,...,n, the auxiliary unit matrix $[v_{ij}] = [\delta_{i\,j+1}]$, where $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ otherwise;

e.g.

$$V = \begin{bmatrix} \cdot & 1 & & \\ & \cdot & 1 & \\ & & \cdot & 1 \\ & & & \cdot \end{bmatrix}.$$

Let $C_{ij}(\lambda_i) = \lambda_i I_{ij} + V_{ij}$,

where I_{ij}, V_{ij} are unit and auxiliary unit matrices of order ν_{ij}, respectively.

By $C = \text{diag}\,[C_{11}(\lambda_1), C_{12}(\lambda_1) \dots, C_{ij}(\lambda_i), \dots C_{rp_r}(\lambda_r)]$

$$\lambda_i \neq \lambda_k, \quad i \neq k,$$
$$j = 1, \dots p_i, \quad i = 1, \dots r,$$

we shall denote the direct sum

$$C_{11}(\lambda_1) \dotplus \quad \dotplus C_{ij}(\lambda_i) \dotplus \dots \dotplus C_{rp_r}(\lambda_r)$$

i.e. a matrix whose diagonal submatrices are the $C_{ij}(\lambda_j)$ and which has zeros elsewhere. It is known (C.f. Turnbull & Aitken, (1932) Wedderburn (1934)), that there is a non-singular matrix Q such that

$$Q^{-1} A Q = C, \tag{5}$$

The latent roots (with their multiplicities) of C being the same as those of A, we have for the λ_i and ν_{ij} of (5),

(α) the λ_i are the distinct latent roots of A,

(β) $\sum_{j=1}^{p_i} \nu_{ij} = \nu_i$, the multiplicity of λ_i.

Let Q be partitioned vertically

$$Q = [Q_{11} | \dots |Q_{ij}| \dots |Q_{rp_r}],$$

so that Q_{ij} has ν_{ij} columns. Let us denote the s-th column of Q_{ij} by x_s^{ij}.

From (5) it follows that

$$AQ = QC,$$

whence, dropping the superfixes and suffixes i, j,

$$A x_s = \lambda x_s + x_{s-1} \quad \text{if } 2 \leq s \leq \nu,$$

and $\quad A\, x_1 = \lambda\, x_1 \quad ;$

or $\quad (A - \lambda I)\, x_s = x_{s-1} \quad \text{if } 2 \leq s \leq \nu \,, \qquad (6)$

and $\quad (A - \lambda I)\, x_1 = 0 \,. \qquad (7)$

Similarly if $U = Q^{-1}$ is partitioned horizontally,

$$U = \begin{bmatrix} U_{i1} \\ \hline \\ U_{ij} \\ \hline U_{r p_r} \end{bmatrix} \qquad \begin{array}{l} j = 1, \ldots, p_i, \\ i = 1, \ldots, r, \end{array}$$

so that U_{ij} has ν_{ij} rows, we obtain from (5)

$$UA = CU \,.$$

Let us denote the s-th row of U_{ij} by $y_s^{ij\,\prime}$, and dropping the superfixes and suffixes i, j, as before, we now have

$$y_s' (A - \lambda I) = y_{s+1}' \,, \quad 1 \leq s \leq \nu - 1 \,, \qquad (8)$$
$$y_\nu'(A - \lambda I) = 0 \,. \qquad (9)$$

α The matrix C, is called the classical canonical form of A. It is unique (apart from a permutation of the order of the $C_{ij}(\lambda_i)$)). This is usually deduced from the properties of the elementary divisors, quantities associated with the classical canonical submatrices $C_{ij}(\lambda_i)$ of A, (Cf. Turnbull & Aitken (1932)). It may however be proved directly. ~~see Appendix~~ .

6.3.

It is easily seen that the rank of $C - \lambda_i I$ is $n - p_i$.

Pre- or postmultiplication by a non-singular matrix does not alter the rank. Hence the rank of is also $n - p_i$. We conclude ~~by~~

$$A - \lambda_i I = Q (C - \lambda_i I) Q^{-1}$$

by 6, 1(e) that the p_i latent vectors x^{ij}_1 , $j = 1, \ldots, p_i$ are the only linearly independent latent vectors associated with λ_i. Similarly, $y^{ij}_{\nu_{ij}}$, $j = 1, \ldots, p_i$, are the only linearly independent latent row vectors associated with λ . (Actually, of course, y^{ij}_1 is a column vector. We shall say " y is a latent row vector", meaning " y' is a latent row vector". This will save putting a dash after the superfixes.) We immediately obtain a well-known result:

<u>Theorem 1.</u> <u>The multiplicit~~ies~~y of the latent root λ_i of A equals the number of latent column (or row) vectors associated with it if and only if all classical canonical submatrices associated with λ_i are of order 1.</u>

6.4.

We shall call the column (row) vectors x^{ij}_s (y^{ij}_s), $s = 1, \ldots, \nu_{ij}$ $j = 1, \ldots, p_i$ " the generalized latent column (row) vectors associated with λ_i ". The vectors x^{ij}_1 $(y^{ij}_{\nu_{ij}})$ will be called "primary latent vectors" and "primary" may be omitted, the vector x^{ij}_s $(y^{ij}_{\nu_{ij}-s+1})$ $1 \leq s \leq \nu_{ij}$, "a latent vector of degree s", in particular we may refer to x^{ij}_2 , x^{ij}_3 $(y^{ij}_{\nu_{ij}-1}$, $y^{ij}_{\nu_{ij}-2})$ as "secondary", "tertiary" latent vectors.

It follows from (6), (7), ((8), (9),) that

$$(A - \lambda_i I)^r x_s^{ij} = 0 \quad , r \geqslant s, \qquad (10)$$

$$y^{ij}_{\nu_{ij}-s+1}(A - \lambda_i I)^r = 0 . \qquad (11)$$

For a 4 x 4 matrix V, an auxiliary unit matrix,

$$V^0 = I ,$$

$$V = \begin{bmatrix} \cdot & 1 & & \\ & \cdot & 1 & \\ & & \cdot & 1 \\ & & & \cdot \end{bmatrix} ,$$

$$V^2 = \begin{bmatrix} \cdot & \cdot & 1 & \\ & \cdot & \cdot & 1 \\ & & \cdot & \cdot \\ & & & \cdot \end{bmatrix} ,$$

$$V^3 = \begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ & \cdot & \cdot & \cdot \\ & & \cdot & \cdot \\ & & & \cdot \end{bmatrix} ,$$

$$V^4 = 0 .$$

More generally for an $n \times n$ matrix V, V^r is of rank $n-r$ if $r < n$, and $V^r = 0$ if $r \geqslant n$. Hence the rank of $(C - \lambda_i I)^r$ or $(A - \lambda_i I)^r$ is $n - \sum_{j=1}^{p_i} q_{ij}$, where $q_{ij} = \min(\nu_{ij}, r)$.

As in 6.3, we deduce that the only latent vectors of $(A - \lambda_i I)^r$ associated with 0 are x_s^{ij} $(y^{ij}_{\nu_{ij}-s+1})$, $s = 1, \ldots \min(\nu_{ij}, r)$, $j = 1, \ldots, p_i$.

Hence any latent vector of degree r is of the form

$$z = \sum_{j=1}^{p_i} \sum_{s=1}^{q_{ij}} c_s^{ij} x_s^{ij}, \tag{12}$$

~~where $x_s^{ij} = 0$, $s \geq \nu_{ij}$~~ and there is a j for which $c_r^{ij} \neq 0$ and

$$y = \sum_{j=1}^{p_i} \sum_{s=1}^{q_{ij}} c^{ij}_{\nu_{ij}-s+1} \, y^{ij}_{\nu_{ij}-s+1}, \tag{13}$$

~~where $y_s^{ij} = 0$, $s \leq 0$~~ and there is a j for which $c^{ij}_{\nu_{ij}-r+1} \neq 0$.

We have from (6) and (8),

$$(A - \lambda_i I) z = \sum_{j=1}^{p_i} \sum_{s=1}^{q_{ij}} c_s^{ij} x_{s-1}^{ij}, \tag{14}$$

where $x_0^{ij} = 0$.

Also $$y'(A - \lambda_i I) = \sum_{j=1}^{p_i} \sum_{s=1}^{q_{ij}} c^{ij}_{\nu_{ij}-s+1} \, y^{ij\,\prime}_{\nu_{ij}-s+2}, \tag{15}$$

where $y^{ij}_{\nu_{ij}+1} = 0$.

Suppose now that Q_1 is any matrix for which $Q_1^{-1} A Q_1 = C$. As Q is non-singular we may put $Q_1 = QT$, where T is non-singular.

From (14)

$$T = \operatorname{diag} \left[T^{c1} \quad T^{c2} \; . \; . \; . \; T^{cr} \right], \tag{16}$$

where T^{ci} is a square matrix of order ν_i. If T^{ci} is partitioned horizontally and vertically so that the matrices in the diagonal are $\nu_{ij} \times \nu_{ij}$, the (ρ, σ)-th component of T^{ci}, $T^{ci}_{\rho\sigma}$, is $\nu_{i\rho} \times \nu_{i\sigma}$. A typical $T^{ci}_{\rho\sigma}$ is

$$T^{ci}_{\rho\sigma} = \begin{bmatrix} \cdot & \cdot & t_3 & t_2 & t_1 \\ \cdot & \cdot & \cdot & t_3 & t_2 \\ \cdot & \cdot & \cdot & \cdot & t_3 \\ & & & & \end{bmatrix}, \tag{17}$$

$$\text{or} \quad T^{ii}_{\rho\sigma} = \begin{bmatrix} t_3 & t_2 & t_1 \\ \cdot & t_3 & t_2 \\ \cdot & \cdot & t_1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}, \qquad (17)'$$

according as $\nu_{i\rho} \leq \nu_{i\sigma}$, or $\nu_{i\rho} \geq \nu_{i\sigma}$. This follows immediately from (14) as the columns and rows of Q_1 are also generalized latent vectors. The conditions (16) and (17) also follow from the commutativity of C and T (Cf. Turnbull & Aitken (1932), Chapter X). The commutativity may be proved thus:

$$C = Q_1^{-1} AQ_1 = T^{-1}Q^{-1} A Q T = T^{-1} C T.$$

We deduce from (15) that T^{-1} also satisfies (16) and (17).

6.5.

Let x^{ij}_s $s = 1, \ldots \nu_{ij}$, $j = 1, \ldots, p_i$, be a set of generalized latent vectors such that the x^{ij}_s are columns of a matrix Q for which $Q^{-1} AQ = C$. Such a set will be called a "complete set of generalized latent vectors associated with λ_i". If z satisfies (12), then z is a latent vector of degree r. It is not to be supposed, however, that any z satisfying (12) is a vector of some complete set. A fortiori, we may have a sequence of generalized latent vectors z_s, $s = 1, \ldots p_i$, $(A - \lambda_i I) z_s = z_{s-1}$, which is not part of a complete set.

To show this assume that z^{ik}_r is of form (12) and that $z^{i\rho}_\sigma$ $\sigma = 1, \ldots \nu_{ij}$, $\rho = 1, \ldots p_i$, is a complete set. Let $c^{ij}_s \neq 0$ in (12) and let the $z^{i\rho}_\sigma$ be ~~form~~ columns of the matrix Q_1. Then $Q_1 = QT$, for some T satisfying (17).

Comparing (12) and QT we see that c_s^{ij} is the element of T_{jk}^{ii} in the s-th row and r-th column.

Hence if $\nu_{ik} \leq \nu_{ij}$ it follows by inspection of T_{jk}^{ii}, that we may choose c_s^{ij} non-zero if and only if

$$r \geq s . \tag{18}$$

If $\nu_{ik} > \nu_{ij}$ we may choose c_s^{ij} non-zero if and only if

$$r \geq \nu_{ik} - \nu_{ij} + s,$$

$$r - \nu_{ik} \geq s - \nu_{ij} . \tag{19}$$

If $\nu_{ik} \leq \nu_{ij}$, (18) implies (19). If $\nu_{ik} > \nu_{ij}$, (19) implies (18). Hence we may choose c_s^{ij} non-zero if, and only if, (18) and (19) are satisfied.

Let us now take a specific example. Let $p_i = 2$, $\nu_{i1} = 1$, $\nu_{i2} = 3$. The vector $x_1^{i1} + x_2^{i2}$ is a latent vector of degree 2, but it is not part of any complete set. For either $k = 1$, $j = 2$ or $k = 2$, and $j = 1$. If $k = 1$, $j = 2$, then $r = 1$, $s = 2$, and (18) is not satisfied. If $k = 2$, $j = 1$, then $r = 2$, $s = 1$, $\nu_{ik} = 3$, $\nu_{ij} = 1$, and (19) is not satisfied. (The invariance of the ν_{ij} is of course a consequence of the uniqueness of C).

6.6

We may see the necessity of (18) and (19) in a different manner. We shall first prove a theorem.

Theorem 2. There is no x such that

$$(A - \lambda_i I)^s x = v + x^{ij}_{\nu_{ij}-r+1} \qquad (20)$$

when $y^{ij\prime}_{\nu_{ij}-r+1} v = 0$, and $s \geq r$.

The vector $y^{ij}_{\nu_{ij}-r+1}$ is a (primary) latent row vector, associated with 0, of $(A - \lambda_i I)^s$ $s \geq r$ and

$$y^{ij\prime}_{\nu_{ij}-r+1} \; x^{ij}_{\nu_{ij}-r+1} = 1 .$$

Hence if (20) were satisfied

$$0 = y^{ij\prime}_{\nu_{ij}-r+1} (A - \lambda_i I)^s x = y^{ij\prime}_{\nu_{ij}-r+1} v + y^{ij\prime}_{\nu_{ij}-r+1} x^{ij}_{\nu_{ij}-r+1}$$
$$= 1,$$

and the theorem follows.

Corollary. There is no vector x such that

$$(A - \lambda_i I) x = x^{ij}_{\nu_{ij}} .$$

This is the theorem for $v = 0$, and $r = 1$.

Now let z^{ik}_r be a latent vector of degree r. It is of form (12), and if $c^{ij}_s \neq 0$ in (12), this equation yields immediately $r \geq s$. Let us now suppose that there is a sequence z^{ij}_t $t = 1, \ldots \nu_{ik}$. By the orthogonality properties of x^{ij}_s, y^{ij}_s, $z^{ik}_r = v + c^{ij}_s x^{ij}_s$ where $y^{ij\prime}_{s*} v = 0$. But

$$(A - \lambda_i I)^{\nu_{ik}-r} z^{ik}_{\nu_{ik}} = z^{ik}_r = v + c^{ij}_s x^{ij}_s ,$$

and hence, by theorem 2,

$$\nu_{ik} - r \quad < \nu_{ij} - s + 1 ,$$

and this is equivalent to (12) .

Analogous results hold for sets of vectors y_s^{ij} $s = 1, \ldots \nu_{ij}$, $j = 1, \ldots p_i$, forming rows of U, where $U A U^{-1} = \mathcal{C}$. Such a set will be called " a complete set of generalized latent row vectors associated with λ_i " .

6.7.

Let us now suppose that the multiplicity of λ_h is known to be ν_h , that there are generalized latent vectors z_s^{hj}, $s = 1, \ldots, \nu'_{hj}$, $j = 1, \ldots p'_h$, and that $\sum_{j=1}^{p'_h} \nu'_{hj} = \nu_h$. Can we assert that the z_s^{hj} are part of a complete set of generalized latent vectors and consequently that there are p'_h classical canonical submatrices of orders $\nu'_{h1}, \nu'_{h2} \ldots \nu'_{hp'_h}$ associated with λ_h? The answer is "yes".

Let x_s^{ij} , $s = 1, \ldots, \nu_{ij}$, $j = 1, \ldots, p_i$, $i = 1 , \ldots, k$, form complete sets. Let Q be partitioned vertically : $Q = [Q_1 \vdots Q_2 \vdots \quad \vdots Q_k]$, where the columns of Q_i are x_1^{i1} , $x_2^{i1}, \ldots$, $x_s^{ij} \ldots$, $x_{\nu_{ip_i}}^{ip_i}$, and let C be partitioned conformably. Then $C = \text{diag} [C_1 , C_2 , \ldots C_k]$. In 6.3. we showed that the z_s^{hj} satisfy (12); in other words the z_s^{hj} are linear combinations of the x_s^{hj} . Hence if $\hat{Q}_h = [z_1^{h1} z_2^{h1} \ldots z_{\nu_{hp_h}}^{hp'_h}]$, $\hat{Q}_h = Q_h T_h$, where T_h is a $\nu_h \times \nu_h$ matrix. As both $Q_h, \hat{Q}_h$ are of rank ν_h (there being ν_h linearly independent x_s^{hj} , z_s^{hj}), T_h is non-singular. We put

$$T = \text{diag}\left[I_1 , I_2 \dots I_{h-1}, T_h , I_{h+1} \dots I_k \right],$$

where I_i is the $\nu_i \times \nu_i$ unit matrix,

and $\hat{Q} = Q \cdot T.$

Since T_h is non-singular, T is non-singular, and it follows that $\hat{Q} = Q\,T$ is non-singular.

Our assumption about the z_s^{hj} is equivalent to

$$A\,\hat{Q}_h = Q\,\hat{C}_h ,$$

where $\hat{C}_h = \text{diag}\left[C_{hj}(\lambda_h) \right]$ $j = 1, \dots p_h'$; C_{hj} of order ν'_{hj}.

Hence if $\hat{C} = \text{diag}\left[C_1, C_2 , \dots C_{h-1}\ \hat{C}_h, C_{h+1} \dots C_k \right],$

$$A\hat{Q} = \hat{Q}C ,$$

or $$\hat{Q}^{-1}\,A\,\hat{Q} = C .$$

By the uniqueness of the classical canonical form, it follows that there is a conjugate permutation of rows and columns of C (denoting the matrix so obtained again by C) such that $\hat{C} = C$. This is the result, by definition of a complete set.

6.8.

We shall now state a result which we require later.

Theorem 3. Let x be a latent column vector of A associated with the latent root λ. There is a latent row vector y' associated with λ, such that $y'x \neq 0$ if and only if one of the classical canonical submatrices of A associated with λ is of order 1.

Since $UQ = I$,

We deduce that

$$y_{s}^{ij\prime} \cdot x_{s'}^{i'j'} = \delta_{ii'} \cdot \delta_{jj'} \cdot \delta_{ss'} \,.$$

Hence $$y_{\nu_{ij}}^{ij\prime} \cdot x_{1}^{ij} = 0 \,,$$

unless some $\nu_{ij} = 1$, say $\nu_{ik_s} = 1$, $s = 1, \ldots t_i$,

when $$y_{\nu_{ik_s}}^{ik_s\prime} \cdot x_{1}^{ik_s} = 1 \,.$$

As $$x = \sum_{j=1}^{p_i} c_i^{ij} x_i^{ij},$$
$$y = \sum_{j=1}^{p_i} d_{\nu_{ij}}^{ij} y_{\nu_{ij}}^{ij},$$
by (12) and (13) respectively,

we have $$y'x = \sum_{s=1}^{t_i} c_1^{ik_s} d_1^{ik_s},$$
and the theorem follows.

6.9.

The principal idempotent and nilpotent elements of A.

Let I_j be the $\nu_j \times \nu_j$ unit-matrix, O_j the $\nu_j \times \nu_j$ null-matrix, and let V_{jk} be the $\nu_{jk} \times \nu_{jk}$ auxiliary unit matrix.

Let $W_j = \text{diag}\,[V_{j1}, V_{j2} \ldots\ldots V_{jp_j}]$.

Put $$I_i = \text{diag}\,[\delta_{ij} I_j], \quad j = 1, \ldots, k,$$
$$= \text{diag}\,[O_1 \ldots\ldots O_{i-1}, I_i, O_{i+1} \ldots, O_k]$$
$$W_i = \text{diag}\,[\delta_{ij} W_j] \quad j = 1, \ldots, k.$$

(The slight inconsistency of notation will cause no confusion).

Let $$E_i = Q I_i Q^{-1} = Q_i I U_i, \tag{21}$$
$$N_i = Q W_i Q^{-1} = Q_i W U_i, \tag{22}$$
where $$Q_i = [\delta_{i1} Q_1 \,\vdots\, \delta_{i2} Q_2 \,\vdots\, \quad \vdots\, \delta_{ik} Q_k],$$
$$U_i' = [\delta_{i1} U_1' \,\vdots\, \delta_{i2} U_2' \,\vdots\, \quad \vdots\, \delta_{ik} U_k'],$$
$$W = [W_1 \,\vdots\, W_2 \,\vdots\, \cdots \vdots\, W_k].$$
~~where Q_i (U_i') is $n \times \nu_i$.~~

Let $$C_i = \lambda_i I_i + W_i;$$

then $$C = \sum_{i=1}^{k} C_i ,$$

whence $$A = Q^{-1} C Q = \sum_{i=1}^{k} Q^{-1} (\lambda_i I_i + W_i) Q = \sum_{i=1}^{k} (\lambda_i E_i + N_i) , \quad (23)$$

$$I = Q^{-1} I Q = \sum_{i=1}^{k} Q^{-1} I_i Q_i = \sum_{i=1}^{k} E_i . \quad (24)$$

We verify immediately

$$N_i = (A - \lambda_i I) E_i ,$$

$$E_i E_j = \delta_{ij} E_j ,$$

$$E_i N_j = \delta_{ij} N_j ,$$

$$N_i N_j = 0 , \qquad i \neq j .$$

As $V_{ij}^{\nu_{ij}} = 0$ and $W_i^{r} = \mathrm{diag}[V_{ij}^{r}]$ $j = 1, \ldots, p_i$,

$$W_i^{s} = 0 , \quad s \geq \nu_{i1} ,$$

while $W_i^{s} \neq 0$, $0 \leq s < \nu_{i1}$, as $V_{ij}^{s} \neq 0$, $s < \nu_{ij}$,

provided that the ν_{ij} are arranged so that $\nu_{i1} \geq \nu_{i2} \geq \quad \geq \nu_{ip_i}$.

Hence $N_i^{s} = 0$, $s \geq \nu_{i1}$,

while $N_i^{s} \neq 0$ $0 \leq s < \nu_{i1}$.

The matrices E_i , N_i are called the principal idempotent and nilpotent elements of A respectively.

The matrix Q is not in general unique, but it is easily proved, that E_i and N_i as defined by (21) and (22) are unique.

For if $$Q_1 A Q_1^{-1} = C ,$$

$$Q_1 = Q T ,$$

where T satisfies (16) and (17).

Hence T commutes with I_i, W_i , (Cf. Turnbull and Aitken, (1932), Chapter X) and

$$Q_1 \; I_i \; Q_1^{-1} = Q\,T\,I_i\;T^{-1}\,Q^{-1} = Q\;I_i\;Q^{-1} = E_i\,,$$

$$Q_1 \; W_i \; Q_1^{-1} = Q\,T\,W_i\;T^{-1}\,Q^{-1} = Q\;W_i\;Q^{-1} = N_i\,.$$

6.10.

In 4.5. we used infinite power series of matrices, e.g. $e^A = \sum_{r=0}^{\infty}(A^r/\,r\,!)$.
We shall now prove some conditions for the convergence of these series, c.f. Turnbull and Aitken (1932), Chapter VI.

When I, V are $n \times n$,

$$f(\lambda I) = f(\lambda)\;I\,,$$

and
$$f(\lambda I + V) = f(\lambda)\,I + f'(\lambda)\,V + \ldots + \frac{1}{(n-1)!}f^{(n-1)}(\lambda)V^{(n-1)},$$
as $V^n = 0$.
Since I_i, W_i, and $C = \sum_{i=1}^{k}(\lambda_i I_i + W_i)$ are direct sums of such matrices (of order ν_{ij}) and null-matrices, it follows that

$$f(C) = f\,(\sum_{i=1}^{k}(\lambda_i I_i + W_i)) =$$
$$= \sum_{i=1}^{k}\left[f(\lambda_i)\,I_i + f'(\lambda_i)\;W_i + \;\ldots\ldots + \frac{1}{(\nu_{i1}-1)!}\,f^{(\nu_{i1}-1)}(\lambda_i)\;W_i^{(\nu_{i1}-1)}\right] \tag{25}$$

and hence $f(A) = Q\,f(C)\,Q^{-1} =$
$$= \sum_{i=1}^{k}\left[f(\lambda_i)E_i + f'(\lambda_i)N_i + \ldots\ldots + \frac{1}{(\nu_{i1}-1)!}\;f^{(\nu_{i1}-1)}(\lambda_i)N_i^{(\nu_{i1}-1)}\right]. \tag{26}$$
Let us suppose that $f(x)$ is a power series. Such a series may be differentiated term by term within its circle of convergence, yielding $f'(x)$, $f''(x)$ ....

No two of the $\sum_{i=1}^{k}\nu_{i1}$ matrices I_i, W_i^s, $s = 1, \ldots.\,\nu_{i1}-1$, $i = 1, \ldots, k$, have a non-zero element in the same position. Hence the series (25) converges if and only if each of its terms exists, i.e. if and only if $f^{(s)}(\lambda_i)$ exists, $s = 0, 1, \ldots \nu_{i1}-1$. As $f(A) = Q\, f(C)\, Q^{-1}$, $f(A)$ exists if and only if $f^{(s)}(\lambda_i)$ $s = 0, 1, \ldots \nu_{i1}-1$ exists. Hence $f(A)$ certainly exists if all latent roots of A are within the circle of convergence of $f(x)$. The series $f(A)$ then satisfies (26).

It follows that

$$e^A = \sum_{r=0}^{\infty} (A/r!)$$

exists for all A as $e^x = \sum_{r=0}^{\infty} (x/r!)$ exists for all x.

Also $\log (I - A) = - \sum_{r=1}^{\infty} (A/r)$

and $$(I - A)^{-p} = I + pA + \{p(p+1)/2!\} A^2 + \ldots + \cdots \qquad (27)$$

exists when $|\lambda_i| < 1$, $i = 1, \ldots, k$.

We finally require to prove that $(I - A)^{-p}$ as defined by (27) is the inverse of $(I - A)^p$ when p is a non-negative integer.

By (26) and (27)

$$(I - A)^{-p} = \sum_{i=1}^{k} \Big[(1 - \lambda_i)^{-p} E_i + p(1-\lambda_i)^{-(p+1)} N_i + \cdots + \frac{(p+\nu_{i1}-2)!}{(p-1)!(\nu_{i1}-1)!}(1-\lambda_i)^{-(p+\nu_{i1}-1)} N_i^{(\nu_{i1}-1)} \Big], \qquad (28)$$

while, by (26)

$$(I - A)^p = \sum_{i=1}^{k} \left[(1-\lambda_i)^p E_i - p(1-\lambda_i)^{(p-1)} N_i + \left\{ \begin{array}{ll} \dfrac{(-1)^{(\nu_{i1}-1)}(1-\lambda_i)^{(p-\nu_{i1}+1)}}{(p-\nu_{i1}+1)!\,(\nu_{i1}-1)!} N^{(\nu_{i1}-1)} & \text{when } p \geq \nu_{i1} \\ (-1)^p N_i^p & \text{when } p < \nu_{i1} \end{array} \right. \right]. \qquad (29)$$

We may obtain $(I - A)^{-p}(I - A)^p = I$, by multiplication of the finite series of (28) and (29).

CHAPTER 7.

7.1.

In this ~~section~~ chapter and chapter 8, we shall be concerned with the basic algebraic properties of (weakly) positive matrices. Most of these are due to Frobenius (1908, 09, 12). We shall prove some of these properties several times, along different lines. The methods we shall use are those of Frobenius, Ostrowski (1937) and Wielandt (1950). We shall not attempt to follow any of these authors in detail. The term "Method of Frobenius" refers to the type of arguments used. Thus Frobenius actually first proved 6.3, Theorem 2, for strictly positive A, then deduced the properties of weakly positive matrices, while we reverse the order. ~~Under "Ostrowski's Method", we shall prove some results not proved by Ostrowski.~~ Finally, we shall give a variant of Wielandt's Method.

7.2.

When A is 1×1 we shall define adj $A = [1]$.
This definition will be convenient in the following sections. As a justification we might add:

If $A = [a]$,

we have $|A| = \cancel{0}\ a$;

and if $a \neq 0$, A has the unique inverse $A^{-1} = [a^{-1}]$.

If $n > 1$, adj $A = |A| \; A^{-1}$, $|A| \neq 0$,

and hence it is natural to define adj $A = [1]$, where A is of order 1.

When $n \geq 2$, adj A is a continuous function of the a_{ij}. Hence we would naturally take adj $[0] = [1]$, making adj $[a]$ a continuous function of a.
We might add that all the properties of the adjoint hold for this definition of adj $[a]$, except those in whose proof the condition $n \geq 2$ has been implicitly used. An example is:

If $|A| \neq 0$,

$$|\text{adj } A| = |A|^{n-1}.$$

~~Hence adj~~

As $|\text{adj } A|$ is a continuous function of the a_{ij},

$$|\text{adj } A| = 0 \quad \text{if } |A| = 0, \quad \text{and } n \geq 2;$$

but $|\text{adj } A| = 1$ if $|A| = 0$, and $n = 1$.

7.3.
Frobenius' Method.

Theorem 1.

(1) Let $A \geq 0$. The matrix A has a non-negative latent root.

(2) If ρ is the largest non-negative latent root of A, and q_i is the corresponding latent root of A_{ii}, the principal minor of A complementary to a_{ii},

then $\rho \geq q_i$.

(3) $\text{adj}(sI - A) \geq 0$, $s \geq \rho$.

(4) $|sI - A| > 0$, $s > \rho$.

Corollary : $(sI - A)^{-1} \geq 0$, $s > \rho$.

(5) The latent root ρ has an associated latent column (row) vector x (y') such that $x > 0$ ($y' > 0$).

(a) The proofs are by induction.

Let $(1)_n$, $(2)_n$,..., denote (1), (2), for matrices of order n. $(1)_1$, $(4)_1$, $(5)_1$, are trivial; thus when $A = [a]$

$$\rho = a \geq 0.$$

$$|sI - A| = s - a > 0 \qquad \text{when} \qquad s > a ,$$

$$x = y = \{1\}.$$

Also, $(3)_1$ is true by our definition, and $(2)_1$ may be taken as $\rho > 0$.

(b) Consider the expansion of $|A|$,

$$|A| = a_{i1}|A_{i1}| + a_{i2}|A_{i2}| + \dots + a_{in}|A_{in}|,$$

where A_{ij} is the signed cofactor of a_{ij} .

Expanding each A_{ij} , $j \neq i$ by elements of the i-th column, we obtain

$$|A| = a_{ii}|A_{ii}| + \sum_{j,k}' a_{ki} \quad a_{ij} \quad |A_{ij}, {}_{ki}| ,$$

where $|A_{ij}, {}_{ki}|$ is the signed cofactor of a_{ki} in the expansion of the signed minor A_{ij} , and $\sum'$ denotes summation over 1 ,..., n. with $i \neq j$, $i \neq k$. With a similar notation $|A_{ii}, {}_{kj}|$ denotes the signed cofactor of a_{kj} in the expansion of $|A_{ii}|$. The minors $|A_{ij},{}_{ki}|$ and $|A_{ii},{}_{kj}|$ consist of the same elements and it is easily proved that

$$|A_{ii},{}_{kj}| = - |A_{ij},{}_{ki}| \quad , j \neq i, k \neq i .$$

Hence $A = a_{ii} \left|A_{ii}\right| - \sum_{j,k}' a_{ki} \; a_{ij} \left|A_{ii,kj}\right|.$

This is the well-known Cauchy expansion of A with respect to the i-th row and column.

We see immediately that the Cauchy expansion may also be written as

$$A = a_{ii} \left|A_{ii}\right| - \hat{a}_{i*} (\operatorname{adj} A_{ii}) \, \hat{a}_{*i},$$

where $\hat{a}_{i*}$, $\hat{a}_{*i}$, denote the i-th row and column of A with a_{ii} omitted. With our definition of the adjoint of a 1 x 1 matrix, the Cauchy expansion also holds for matrices of order 2, in this form.

(c) Let us now assume $(1)_{n-1}$, $(2)_{n-1}$ ...., $n > 1$. ~~Now~~ Let $A(s) = |sI - A|$, $A_{ii}(s) = |sI_{n-1} - A_{ii}|$, where I_{n-1} is the unit matrix of order $n - 1$.

Then $A(s) = (s - a_{ii}) \; A_{ii}(s) - \hat{a}_{i*} \operatorname{adj} (s I_{n-1} - A_{ii}) \, a_{*i}$.

By $(1)_{n-1}, (3)_{n-1}$ there is a $q_i \geq 0$ such that

$$A_{ii}(q_i) = 0,$$

and $\operatorname{adj} (q_i I_{n-1} - A_{ii}) \geq 0$.

Hence $A(q_i) = - \hat{a}_{i*} \operatorname{adj} (q_i I_{n-1} - A_{ii}) \hat{a}_{*i} \leq 0$, as $\hat{a}_{i*} \geq 0$, $\hat{a}_{*i} \geq 0$.

By a diagonal expansion of $A(s)$ (Cf. 6.1g) we obtain

$$A(s) = |sI - A| = \sum_{k=0}^{n} t_{n-k}(q_i) \, (s - q_i)^k,$$

where $t_{k}(q_i)$ is the sum of principal minors of order k of $(q_i I - A)$, and $t_0(q_i) = 1$.

The term of the highest degree will dominate the others when s is large.
Hence $A(s) > 0$ for sufficiently large s.
As $A(s)$ is continuous there is a ρ such that

$$A(\rho) = |\rho I - A| = 0,$$

for we have proved $A(q_i) \leq 0$ for all i.
This proves $(1)_n$.

If we take ρ to be the largest non-negative latent root then certainly $\rho \geq q_i \quad i = 1, 2, \ldots, n$.
Hence $(2)_n$ holds and $(4)_n$ follows.

(d) Expanding A_{ij} by elements of the j-th row and we obtain

$$|A_{ij}| = \sum_k{}' a_{jk} \, |A_{ij,jk}|$$
$$= - \sum_k{}' a_{jk} \, |A_{jj,ik}|,$$

Where $\sum_k{}'$ denotes summation from $1, \ldots, n$ with $k \neq j$.
When $i \neq j$, and $n \geq 3$,

$$|A_{ij}| = - a_{ji} \, |A_{jj,ii}| - \sum_{k,l}{}'' a_{jk} \, a_{li} \, |A_{jj,ik,li}|$$
$$- a_{ji} \, |A_{jj,ii}| + \sum_{k,l}{}'' a_{jk} \, a_{li} \, |A_{jj,ii,lk}|,$$

where $\sum_{k,l}{}''$ denotes summation with $k, l \neq i,\ k, l \neq j$.
Hence

$$|A_{ij}| = - a_{ji} \, |A_{jj,ii}| + \tilde{a}_{j*} \, (\mathrm{adj} \; A_{jj,ii}) \, \tilde{a}_{*i},$$

where $\tilde{a}_{j*}$, $\tilde{a}_{*i}$ denote the j-th row, i-th column of A, with a_{ji}, a_{jj} and a_{ji}, a_{ii} omitted.

Let $\quad sI - A = B$

and $\quad B_{jj,ii} = C$.

Then, if $i \neq j$, $n \geq 3$,

$$|B_{ij}| = + a_{ji} |C| + \tilde{a}_{j*} \text{ adj } C \; \tilde{a}_{*i} ,$$

while $B_{ij} = + a_{ji}$, if $n = 2$. By $(4)_{n-2}$ and $(3)_{n-2}$ it follows that $|B_{ij}| \geq 0$ when s is larger or equal to the largest non-negative latent root of $A_{jj,ii}$ (which may be taken to be 0 if $n = 2$).

Hence,

$$|B_{ij}| \geq 0 \quad \text{when} \quad s \geq \rho ,$$

as by $(2)_n$, $(2)_{n-1}$, ρ is greater or equal to the largest non-negative latent root of $A_{jj,ii}$.

But $|B_{ii}| \geq 0$ when $s \geq \rho \geq q_i$ by $(4)_{n-1}$,

and hence

$$\text{adj } B = [|B_{ji}|] \geq 0 , \quad \text{when } s \geq \rho .$$

This proves $(3)_n$.

(e)

Finally we prove $(5)_n$.

Let $\quad M = \rho I - A$.

Then $|M| = 0$, and hence there is a y' such that

$$y' M = 0 , \qquad y' \neq 0 .$$

Let $\quad y_i \neq 0$.

By $(2)_n$, $q_i \leq \rho$.

Suppose $q_i = \rho$.

Then $|M_{ii}| = 0$, and by $(5)_{n-1}$ there is an $\hat{x} \supset 0$ such that

$$M_{ii}\, \hat{x} = 0 .$$

Let x be the vector for which $x_i = 0$, and such that $\hat{x}$ is obtained from x by omitting x_i.

Let $M\,x = z$.

Then $z_k = \sum_{j=1}^{n} m_{kj}\, x_j = \sum_{j \neq i}^{n} m_{kj} x_j = (M_{ii}\, \hat{x})_k = 0$, $k \neq i$.

But $y' M\, x = \sum_{j=1} y_j\, z_j = y_i\, z_i = 0$,

and as $y_i \neq 0$, $z_i = 0$.

Hence $M\,x = z = 0$, where $x \supset 0$.

Suppose $q_i < \rho$.

Then $|M_{ii}| \neq 0$. By $(3)_n$, adj $M \geqq 0$, and hence adj $M \supset 0$ and indeed, the i-th column of adj M is (weakly) positive. Let x be that column.

As M adj $M = 0$,

it follows that $M\,x = 0$, where $x \supset 0$.

~~This~~ Similarly we may now prove that there is a latent row vector $y' \supset 0$ associated with ρ. This proves $(5)_n$.

The theorem follows by induction.

7.4.

When A is irreducible the results of theorem 1 may be sharpened. It is simplest to treat the case $A \supset 0$ separately, and for the sake of completeness we shall state the corresponding results for the only irreducible null-matrix, the 1 x 1 null-matrix.

Theorem 2. If A is the 1×1 null-matrix

(1) A has the latent root 0,

(3) $|sI - A| > 0$ when $s > 0$,

(4) $\operatorname{adj} A = [1] > 0$,

(5) The latent root 0 has associated with it the positive latent column vector $x = \{1\}$ row vector, $y' = [1]$.

The proofs are trivial.

Theorem 3. Let A, $A \geqq 0$, be irreducible. Then

(1) The matrix A has a positive latent root,

(2) If ρ is the largest positive latent root of A, and q_{ii} is the largest non-negative latent root of A_{ii}, (Cf. Theorem 1, (2)), then $\rho > q_{ii}$,

(3) $\operatorname{adj}(sI - A) > 0$ when $s \geqq \rho$,

(4) $|sI - A| > 0$ when $s > \rho$.

Corollary. $(sI - A)^{-1} > 0$ when $s > r$.

(5) The latent root ρ has an associated latent column vector x, (row vector y') such that

$x > 0$ ($y' > 0$).

(a) We shall prove (5) first, for $\rho \geqq 0$. By theorem 1 (5), the largest non-negative latent root ρ of A has a (weakly) positive latent column vector. Hence (5)' implies (5) with $\rho \geqq 0$, where (5)' is (5): "If x is a positive latent vector associated with the largest non-negative latent root of an irreducible $A \geqq 0$, then x is strictly positive."

Let $x > 0$ be associated with ρ. If $x \not> 0$, then by a conjugate permutation of rows and columns of A_ρ and x, we may arrange the x_i so that $x = \{x^1 \quad x^2\}$ where $x^1 > 0$ and $x^2 = 0$.

Partition $M = \rho I - A$ conformally with x (after the conjugate permutation has been carried out) thus,

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where M_{11}, M_{22} are square matrices.

Now $\quad M x = 0$,

and so $\quad M_{21} x^1 = 0$.

But $\quad M_{21} \leq 0$, and $x^1 > 0$.

and this implies $\quad M_{21} = 0$, by 4.2.7.

But M is irreducible, since the reducibility of a matrix is unaffected when the diagonal elements and the sign are altered. Hence we have a contradiction.

It follows that $x > 0$.

Similarly $\quad y' > 0$, when $y' M = 0$.

We have proved (5).

(b) As $x > 0$ and $A \supset 0$ is irreducible,

$$\rho x = A x > 0,$$

by 4.2.10.

Hence $\rho > 0$.

This proves (1).

(c) By theorem 1 (2), $q_i \leq \rho$.

Suppose $q_i = \rho$. Then $|M_{ii}| = 0$.

When $|M_{ii}| = 0$ we have shown in the proof of theorem 1 (e), that there is a latent vector of A, w, say, associated with ρ such that $w \supset 0$ and $w_i = 0$. This is not possible by $(5)'$. Hence $q_i < \rho$.

We have proved (2).

(d) The result (3) follows immediately from theorem 1, (3).

(e) It remains to prove (4).

If $n = 1$, (4) is trivially true by our definition of 6.2.

Let $B = sI - A$.

By theorem 1, (4) $\text{adj } B \supseteq 0$, when $s \geq \rho$.

If $n > 1$, then by (2) $q_i < \rho$.

Hence, it follows, from (3) applied to A_{ii}, that

$$|B_{ii}| > 0 \qquad \text{when} \qquad s \geq \rho .$$

Thus all diagonal elements of adj B are positive.

There is a conjugate permutation of rows and columns on B and adj B such that z, the i-th column of B after the permutation is $z = \{z^1, z^2\}$ where $z^1 > 0$, $z^2 = 0$, if some $z_j = 0$, since $z \geq 0$; and $z_i > 0$, as z_i is a diagonal element of adj B.

But $Bz = |B|\delta$, where $\delta = \{\delta_{ij}\}$ $j = 1, \ldots, n$.

Let B be partitioned conformably with z, and let $\delta = \{\delta^1 \; \delta^2\}$, conformably with B.

As z_i is an element of z^1 it follows that $\delta^2 = 0$.

~~as z_{in} is a diagonal element of adj B~~

Hence $B_{21} z^1 + B_{22} z^2 = B_{21} z^1 = 0$.

But $B_{21} \leq 0$, and $z^1 > 0$, and hence by 4.2.7, $B_{21} = 0$.
It follows that B, and hence also A, is reducible contrary to assumption.

Hence $z > 0$. We deduce that any column of adj B is strictly positive.

Hence $\operatorname{adj}(sI - A) = \operatorname{adj} B > 0, \quad s \geq \rho$.

This proves (4) and completes the proof of the theorem.

A strictly positive matrix is of course irreducible. Theorem 3 therefore applies to a strictly positive matrix.

CHAPTER 8.

8.1

We shall now digress from our plan of proving the fundamental properties of positive matrices, to state and prove a theorem due to Frobenius (1908) and Collatz (1942). Wielandt's method is closely connected with Collatz's theorem.

Let $e = \{1, 1, \ldots, 1\}$

and $\underline{r} = A e$, or $r_i = \sum_{j=1}^{n} a_{ij}$.

Let $r = \min_i (A e)_i = \min_i \left(\sum_{j=1}^{n} a_{ij}\right)$,

and $R = \max_i (A e)_i = \max_i \left(\sum_{j=1}^{n} a_{ij}\right)$,

and let ρ be the greatest non-negative latent root of $A \geq 0$. It was known to Frobenius that

$$r \leq \rho \leq R \quad . \qquad (1)$$

He proved this for first $A > 0$, and deduced the general result by continuity arguments. We may use his method to prove (1) for ~~A or~~ irreducible $A \geq 0$.

By 6.3, Theorem 3 7.4 T.3

$$\operatorname{adj} (\rho I - A) > 0 . \qquad (2)$$

Let $F = (\rho I - A) K$,

where

$$K = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{bmatrix} .$$

Then $|F| = |\rho I - A| = 0$,

$f_{i1} = \rho - r_i$,

$|F_{i1}| = |B_{i1}|$, where $B = \rho I - A$.

Hence

$$(\rho - r_1)|B_{i1}| + (\rho - r_2)|B_{i2}| + \ldots + (\rho - r_n)|B_{in}| = 0,$$

and $\qquad B_{i1} > 0 \qquad$ by (2) .

If some $(\rho - r_i)$ is positive, another, say $(\rho - r_j)$ is negative.

Hence (1) follows, and further if no $(\rho - r_i)$ is positive, $\rho - r_i = 0$, $i = 1, \ldots, n$. Hence $\rho = R$ or (similarly) $\rho = r$,

if and only if

$$r = \rho = R, \tag{3}$$

$$\text{i.e.} \quad \underline{r} = \rho e .$$

8.2.

Collatz's theorem is a generalisation of (1). Collatz (1942) proved the theorem for $A \supset 0$. We shall consider here the theorem only for irreducible weakly positive A,

~~and return to the general case~~

Let the generalised vector of row sums of A with respect to the diagonal matrix X , $x_i > 0$, be defined by

$$\underline{r} = X^{-1} A X e . \tag{4}$$

The latent roots of $X^{-1} A X$ are those of A , and hence

$$\text{if } r = \min_i r_i ,$$

$$R = \max_i R_i , \quad \text{then}$$

(1) and (3) hold for the generalised $\underline{r}$ of (4) .

Another simple proof is given below.

Let $\quad x > 0$

and $\quad r_i = (A\,x)_i \,/ x_i = (\sum_{j=1}^{n} a_{ij}\, x_j) / x_i\,.$ (5)

The definitions (4) and (5) are equivalent, as we may see by putting $X\,e = x$ in (4) .

Let y' be the strictly positive latent row vector associated with ρ .

$$r_i\, x_i = \sum_{j=1}^{n} a_{ij}\, x_j \,,$$

$$\sum_{i=1}^{n} r_i y_i\, x_i = \sum_{i,j=1}^{n} y_i a_{ij}\, x_j = \rho \sum_{i=1}^{n} y_i\, x_i\,.$$

Hence $\quad \rho = (\sum_{i=1}^{n} r_i\, y_i\, x_i) / (\sum_{i=1}^{n} y_i\, x_i)$

as $\quad \sum_{i=1}^{n} y_i\, x_i = y'\, x > 0\,.$

Hence ρ is a weighted mean of the r_i , $i = 1, \ldots, n$ with weights $y_i\, x_i$. As all weights are positive, (1) follows, and (3) holds if and only if all r_i are equal, i.e., $r = \rho = R$.

If (3) holds,

$$(A\,x)_i \,/\, x_i = \rho\,,$$

whence $\quad A\,x = \rho\, x\,,$

i.e. we have $\rho = R$ or $\rho = r$ if and only if x is the unique positive latent column vector associated with ρ (c.f. 8. 8) .

8.3.

We shall now prove a similar result, which we shall call Collatz's theorem for $x \supset 0$.

Let $x \supset 0$ and let r_i be defined by (5), (5)', (5)'';

$$r_i = (A\,x)_i / x_i\,, \tag{5}$$

where $r_i = \infty$ if $(A\,x)_i > 0\,, \quad x_i = 0\,,$ (5)'

r_i does not exist if $(A\,x)_i = x_i = 0.$ (5)''

Let r and R be defined as on ~~p~~ 8.2, i.e.

as $\quad r = \min r_i\,,$

$\qquad R = \max R_i\,,$

where the minimum (maximum) is taken over the existing r_i.

Suppose $x_i > 0\,, \quad i \in G\,,$

and $\quad x_i = 0\,, \quad i \in E - G\,,$

where E is the set $1, 2, \ldots, n$ and G is a proper subset of E.

Let $A_{21} : [a_{ij}] \quad i \in E - G,\ j \in G\,.$

As A is irreducible, $A \supset 0$. Hence there is a positive a_{ij},

$i \in E - G\,,\ j \in G\,,$ say $a_{kl} > 0$. As $l \in G\,,\quad x_l > 0.$

Now $(A\,x)_k = \sum_{j=1}^{n} a_{kj}\,x_j \geq a_{kl}\,x_l > 0\,,$

and $x_k = 0\,,$ as $k \in E - G\,.$

Hence $\quad r_k = \infty$.

Hence, if, $x \supset 0$, but some $x_i = 0$,

then $\quad R = \infty\,.$ (6)

As some $x_i > 0$, say $x_i > 0$, r_i is finite. Hence $r \leq r_i$ is finite.

If $x_i > 0$ then $r\, x_i \leq r_i\, x_i = (A\, x)_i$.

If $x_i = 0$ then $r\, x_i = 0$ and $(A\, x)_i \geq 0$,

while $(A\, x)_k > 0$ and $r\, x_k = 0$.

Hence $r\, x \subset A\, x$. It follows that

$$r\, y'x < y'A\, x = \rho\, y'x \text{ , when } y' \text{ is the}$$

vector of 8.2.

Thus $r < \rho$.

Hence if $x \supset 0$, some $x_i = 0$ and A is irreducible, then $r < \rho < R = \infty$.

8.4. Wielandt's Method.

Wielandt (1950) bases his proof of the existence of a non-negative latent root of the weakly positive matrix A on a converse of Collatz's theorem for $x \supset 0$, (and $x > 0$).

Let $x \supset 0$ and $x'e = 1$, $e = \{1, 1, \ldots, 1\}$.
Wielandt defines (as in (5))

$$r_i\,(x) = \frac{(A\,x)_i}{x_i} = \frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i}, \qquad (5)$$

where $r_i\,(x) = \infty$, when $x_i = 0$. $\qquad (5)'''$

We shall however replace $(5)'''$ by the definition of $(5)'$ and $(5)''$:

$$r_i\,(x) = \infty \text{ if } (A\,x)_i > 0, \quad x_i = 0, \qquad (5)'$$

and $r_i(x)$ does not exist if $(Ax)_i = x_i = 0$. (5)''

Let $\rho(x) = \min_i r_i(x)$,

where the minimum is taken over the existing $r_i(x)$.

If $x_M = \max_i x_i$, $x_m = \min_i x_i$, $R' = \max_i \sum_{j=1}^{n} a_{ij}$,

then $r_M(x) = \left(\sum_{j=1}^{n} a_{Mj} x_j / x_M\right) \leq (x_M R')/x_M \leq R'$.

Hence $\rho(x)$ possesses a finite upper bound ρ.

From $r_i(x) \geq 0$ it follows that $\rho \geq 0$.

Wielandt then argues : "As the set of vectors $x > 0$, $x'e = 1$, is bounded and closed, the upper bound ρ is attained". This argument leaves a gap. It should be noted that $\rho(x)$ is not a continuous function of x, e.g.

$$A = \begin{bmatrix} 2 & \cdot \\ \cdot & 1 \end{bmatrix},$$

, a fair example, as we have not yet used the condition that A is irreducible, $x = \{x_1, x_2\}$,

$\rho(x) = 1$; unless $x_2 = 0$, when $\rho(x) = 2$:

or, for irreducible A,

$$A = \begin{bmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 2 \\ 1 & \cdot & 2 \end{bmatrix}, \quad x = \{x_1, x_2, x_3\},$$

$r_1(x) = x_2/x_1$, $r_2(x) = 2x_3/x_2$, $r_3(x) = (x_1 + 2x_2)/x_3$.

If $x_1 = x_2 < x_3$, $\rho(x) = r_1(x) = 1$,

unless $x_1 = x_2 = 0$, when $\rho(x) = r_3(x) = 2$.

However, $\rho(x)$ is an upper semi-continuous function of x, i.e.

for any $\epsilon > 0$ we may find a vector $k > 0$ such that

$$f(x+h) \leq f(x) + \epsilon \tag{7}$$

for all h such that $|h_i| \leq k_i \quad i = 1, \ldots, n$.

A function of this type attains its upper bound over a closed bounded region. (I am grateful to Dr. J. Cossar for pointing this out to me.)

We shall prove these statements. The functions of x_i, $(Ax)_i$ and x are continuous. Hence we have from (5) that $r_i(x)$ is continuous, provided that $x > 0$.

Now suppose

$$x_i > 0, \qquad i = 1, \ldots, k,$$

$$x_i = 0, \qquad i = k+1, \ldots, n.$$

Provided that $|h_i| < x_i$, $\quad i = 1, \ldots, n,$

$$x_i + h_i > 0, \qquad i = 1, \ldots, k.$$

If $\quad (x+h) > 0, \quad h_i \geq 0, \quad i = k+1, \ldots, n.$

Suppose $\quad (x_i + h_i) > 0, \qquad i = k+1, \ldots, l$

$$(x_i + h_i) = 0 \qquad i = l+1, \ldots, n.$$

~~Then~~ But $f(x) = \min\limits_{i \leq k} r_i(x)$

as $r_i(x) = \infty$, or does not exist if, $k < i \leq n$.

Now $\quad f(x+h) \leq \min\limits_{i \leq l} r_i(x+h) \leq \min\limits_{i \leq k} r_i(x+h) \leq f(x) + \epsilon$

for sufficiently small h_i, by the continuity of $r_i(x)$, $i \leq k$.

This proves upper semi-continuity.

Now let ρ be the upper bound of $\rho(x)$.
There is a sequence $\{x^k\}$, of vectors $x^1, x^2, \ldots,$ (possibly containing only one distinct vector) such that

$$\rho(x^k) \to \rho, \quad \text{as} \quad k \to \infty .$$

As the set of vectors $x \geq 0$, $x'e = 1$ is bounded and closed, the sequence has a cluster point x^*, and there is a subsequence $\{y^k\}$ of $\{x^k\}$ such that

$$y^k \to x \quad \text{as} \quad k \to \infty .$$

Given $\epsilon > 0$, there are k_0, k_1, satisfying

$$\rho(y^k) \geq \rho - \epsilon \quad \text{for} \quad k \geq k_0, \text{ as } \rho(y^k) \to \rho,$$

and

$$\rho(y^k) \leq \rho(x) + \epsilon, \quad \text{for} \quad k \geq k_1,$$

by upper continuity .

Hence $\rho(x) \geq \rho - 2\epsilon$,
and as $\rho(x)$ is independent of ϵ,

$$\rho(x) = \rho .$$

8.5.

The fact that $\rho(x)$ attains its upper bound ρ is of considerable importance, for we shall show that ρ is the greatest latent root of the irreducible matrix A, while x, $\rho(x) = \rho$, is the latent column vector associated with ρ.

Following Wielandt, we proceed as follows.
Let A be irreducible, and $z \geq 0$. If some $z_i = 0$ we

may apply a conjugate permutation of rows and columns to A and z so that $z = \{z_1, z_2\}$ where $z_1 > 0$, $z_2 = 0$ (z_1, z_2 being vectors).

Let $z^p = (I + A)^{p-1} z$.

Then if I, A, z^2 are partitioned conformally with x,

$$z_1^2 = (I_{11} + A_{11}) z_1 + A_{12} z_2$$

$$= (I_{11} + A_{11}) z_1 \geq I_{11} z_1 > 0,$$

$$z_2^2 = A_{21} z_1 + A_{22} z_2 = A_{21} z_1 \supset 0,$$

by 4.2.6 as $A_{21} \supset 0$, since A is irreducible.

Hence $(I + A) z$ contains at least one more positive element than z.

By repetition of this process, it follows that $(I + A)^{n-1} z > 0$.

Now let $\rho(x) = \rho$ and

$$z = (A - \rho I) x .$$

As
$$\rho(x) = \min_i \frac{(A x)_i}{x_i},$$

$$z = (A - \rho I) x \geq 0 .$$

If $z \supset 0$

$$(I+A)^{n-1} z > 0,$$

$$(I + A)^{n-1} z = (A - \rho I)(I + A)^{n-1} x > 0 .$$

Thus $A (I + A)^{n-1} x > \rho (I + A)^{n-1} x$,

whence
$$\rho < \min_i \frac{(A (I + A)^{n-1} x)_i}{((I + A)^{n-1} x)_i} .$$

But this is impossible by definition of ρ, as $(I + A)^{n-1} x > 0$.

Hence $z = 0$ and

$$A\, x = \rho\, x .$$

This proves that ρ is a latent root of A, and that x is an associated latent vector.

As $x \geq 0$, $(I + A)^{n-1} x > 0$,

but $(I + A)^{n-1} x = (1 + \rho)^{n-1} x$.

Hence $x > 0$.

Thus $A\, x > 0$, ~~(unless A is the 1x1 null matrix)~~

whence $\rho > 0$.

8.6. A Variant of Wielandt's Method.

We may avoid an appeal to the properties of semi-continuous functions by defining a scalar $k > 0$ such that we need only consider $x \geq e/k > 0$, $x'e = 1$. We could then use the argument of 8.5, but we shall replace it by a simpler one.

We shall now consider, not

$$\max_x \left\{ \min_i \; (A\, x)_i / x_i \right\},$$

but

$$\min_x \left\{ \max_i \; (A\, x)_i / x_i \right\}.$$

This could also have been done in 8.4 and 8.5. Some difficulties mentioned in this approach by Wielandt (1950) are easily overcome in 8.8.

Let $x > 0$, and let

$$\rho^*(x) = \max_i (Ax)_i / x_i = \max_i r_i(x),$$

and $\rho^* =$ lower bound of $\rho^*(x)$,

when $x > 0$ and $x'e = 1$.

We have $\rho^*(e/n) = \max_i \sum_{j=1}^{n} a_{ij} = R'$, say,

and $(e'/n)e = 1$.

Hence $\rho^* \leq R'$ where $R' > 0$.

We may choose x so that

$$\rho(x) \leq U \tag{8}$$

for some $U \geq R'$,

and suppose a conjugate permutation of rows and columns carried out on the irreducible matrix A, so that

$$x_1 \geq x_2 \geq x_3 \geq \dots \geq x_n > 0.$$

Then $U \geq \sum_{j=1}^{k} \frac{a_{ij} x_j}{x_i} + \sum_{j=k+1}^{n} \frac{a_{ij} x_j}{x_i}$

$$\geq \frac{x_k}{x_{k+1}} \left(\sum_{j=1}^{k} a_{ij}\right) + a_{ii}, \text{ if } k < i \leq n.$$

Hence by adding these inequalities for $i = k+1, \dots, n$,

$$(n-k)U \geq \frac{x_k}{x_{k+1}} \left(\sum_{i=k+1}^{n} \sum_{j=1}^{k} a_{ij}\right) + \sum_{i=k+1}^{n} a_{ii},$$

whence $\frac{x_k}{x_{k+1}} \leq \frac{(n-k)U - t_2}{s_{21}} = \sigma_k'$, say, (9)

where $s_{21} = \sum_{i=k+1}^{n} \sum_{j=1}^{k} a_{ij} > 0$, as A is irreducible

and $t_2 = \sum_{i=k+1}^{n} a_{ii}$. ~~as A is irreducible~~

We may check , using $U \geq R'$,

$$U - a_{ii} \geq \sum_{j \neq i} a_{ij} \geq \sum_{j=1}^{k} a_{ij} \quad , \quad k < i \leq n ,$$

whence $(n-k)U - t_2 \geq S_{21}$.

Hence $1 \leq \sigma_k'$, a result which is also a trivial consequence of $x_k \geq x_{k+1}$.

Now $\frac{x_k}{x_n} = \frac{x_k}{x_{k+1}} \cdot \frac{x_{k+1}}{x_{k+2}} \cdot \ldots \cdot \frac{x_{n-1}}{x_n} \leq \sigma_k' \sigma_{k+1}' \ldots \sigma_{n-1}'$

$= \Delta'_{kn}$, say .

But $1 = \sum_{i=1}^{n} x_i \leq (\Delta'_{1n} + \Delta'_{2n} + \quad + \Delta'_{nn}) \, x_n$,

where $\Delta'_{nn} = 1$.

Hence $x_n \geq 1/K'_n > 0$,

where $K'_n = \sum_{i=1}^{n} A_{in}$.

It follows that

$$\rho^* = \text{lower bound } \rho^*(x) ;$$

where x satisfies $x \geq e/K'$, $x'e = 1$.

This set is bounded and closed, while $0 < \rho^* \leq R'$. ~~(unless A is the 1 x 1 null matrix)~~. (The relation $0 < \rho^*$ holds as, for the irreducible matrix $A \supset 0$,

$$r_i(x) = \Big(\sum_{j=1}^{n} a_{ij} x_j\Big) / x_i > 0 .)$$

Hence there is a vector x such that

$$\rho^*(x) = \rho^* , \tag{10}$$

as $\rho^*(x)$ is continuous over the set.

8.7.

Let (10) be satisfied .

There is some $r_i(x) = \rho^*$.

Suppose there is a j for which $r_j(x) < \rho^*$.

Now let $r_i(x) = \rho^*$ if $i \in G$,

and $r_i(x) < \rho^*$ if $i \in E - G$,

where G is a non-~~null~~ -empty subset of $E = (1, 2, \ldots, n)$.

As $r(h\,x) = r(x)$, h a positive scalar,

we may omit the condition $x'e = 1$.

If $E - G$ is non-~~null~~ empty, define a vector $\bar{x} > 0$, where

$$\bar{x}_i = x_i \quad , \quad i \in G,$$

$$\bar{x}_i = \alpha\, x_i \quad , \quad i \in E - G,$$

and α satisfies

$$\max_{i \in E-G} \{ r_i(x)/\rho^* \} < \alpha < 1 .$$

When $i \in G$, $r_i(\bar{x}) = (1/x_i)\left(\sum_{j \in G} a_{ij} x_j + \alpha \sum_{j \in E-G} a_{ij} x_j\right)$

$$\leq r_i(x) = \rho^* . \qquad (11)$$

When $i \in E - G$, $r_i(\bar{x}) = (1/\alpha\, x_i)\left(\sum_{j \in G} a_{ij} x_j + \alpha \sum_{j \in E-G} a_{ij} x_j\right)$

$$\leq r_i(x)/\alpha < \rho^* . \qquad (12)$$

Hence $\rho(\bar{x}) \leq \rho^*$.

As A is irreducible, there is an $i \in G$, $j \in E - G$, for which $a_{ij} > 0$. For this i,

$$r_i(x) = (1/x_i)\left(\sum_{j \in G} a_{ij} x_j + \alpha \sum_{j \in E-G} a_{ij} x_j\right) < r_i(x) = \rho^* . \qquad (13)$$

Let $i \in G'$, if $r_i(\bar{x}) = \rho^*$.

By (11), (12), (13), G' contains at least one member less than G. Constructing similarly sets G'' from G', G''', etc., we see that $G^{(n-1)}$ is empty. Hence there is a vector $z > 0$, such that $\rho^*(z) < \rho^*$.

This is impossible, by definition of ρ^*, and hence $G = E$, or

$$(A\,x)_i / x_i \;=\; r_i(x) \;=\; \rho^* , \qquad i = 1, \ldots, n .$$

Hence $A\,x = \rho^*\,x$.

Thus ρ^* is a latent root, and $x > 0$ an associated latent vector.

8.8.

We shall now prove that the latent root ρ of 8.5 (or ρ^* of 8.7) is not exceeded in modulus by any latent root of the (irreducible) matrix A, and that ρ is a single latent root.

(i) Let λ be any latent root of A, and let z be its associated latent row vector;

$$z'A = \lambda\, z' .$$

Therefore $\quad z^{*\prime}A \geq |\lambda|\, z^{*\prime},$

by 4.3. 14 .

Thus $\quad \rho\, z^{*\prime}x = z^{*\prime}A\,x \geq |\lambda|\, z^{*\prime}x$, where x is the positive latent vector of 8.5.

As $z \neq 0$, $z^* > 0$, and so $z^{*\prime}x > 0$, by 4, 2.6.

Hence $\quad \rho \geq |\lambda|$.

A similar argument yields $\quad \rho^* \geq |\lambda|$.

Putting $\lambda = \rho^*$ in the first inequality, $\lambda = \rho$ in the second we obtain

$$\rho = \rho^* .$$

This proves the first part of the statement.

To prove the second part, let us first suppose that there are two latent vectors x, z associated with ρ, and $x > 0$. The vector z is real. There is a c such that

$$w = x - c\, z \geq 0 \quad \text{and some element of } w \text{ is zero.}$$

If $w > 0$, w is also a latent vector associated with ρ, and some $w_i = 0$. This is impossible by 7.4, Theorem 3 (5). Hence $w = 0$, and z and x are linearly dependent. There is only one linearly independent latent vector associated with ρ.

The transposed matrix A' is also positive and irreducible. Hence it possesses a strictly positive latent column vector y associated with its positive latent root of maximum modulus. As the latent roots of A and A' are the same, this latent root equals ρ.

Thus $\quad y' A = \rho\, y'$

and $\quad y' x > 0$.

It follows from 6.8, Theorem 3 that there is a classical canonical submatrix of order 1 associated with ρ. But ρ has only one associated latent vector, and therefore has only one classical canonical submatrix associated with it.

It follows that ρ is a single latent root.

(ii) We have proved in this chapter the existence of a positive latent root of largest modulus for an irreducible A. One may quickly deduce that every $A > 0$ has a non-negative latent root of largest modulus.

Let $\bar{A} : [\bar{a}_{ij}]$ be obtained from A by a conjugate permutation of rows and columns. There is an arrangement $(h_1, h_2, \ldots, h_n)$ of $(1, 2, \ldots, n)$ for which $\bar{a}_{ij} = a_{h_i h_j}$.

Let $Q = [q_{ij}]$ be the matrix satisfying

$$q_{ih_i} = 1, \quad q_{ij} = 0 \quad , \quad j \neq h_i .$$

It is easily proved that

$$Q' \; Q = I ,$$

and
$$Q^{-1} \; AQ = Q' \; AQ = \bar{A} .$$

Hence the latent roots of $\bar{A}$ are those of A. Let us suppose that $\bar{A}$ is in the normal form of 3.7.

Then
$$|\bar{A} - \lambda I| = \prod_{i=1}^{k} |\bar{A}_{ii} - \lambda I_i| ,$$

where I_i is of the same order as $\bar{A}_{ii}$

It follows that the latent roots of A and $\bar{A}$ are those of the k irreducible $\bar{A}_{ii}$ taken together.

But $\bar{A}_{ii} \supset 0$, unless it is the 1×1 null-matrix, whose only latent root is 0.

Hence $\bar{A}_{ii}$ has a non-negative latent root of greatest modulus, ρ_i, say and $\rho = \max_i \rho_i$ is a non-negative latent root of greatest modulus of A.

We deduce that $\rho = 0$, if and only if each $\bar{A}_{ii}$ is an 1×1 null-matrix. In other words the greatest non-negative latent root of A is positive unless $a_{ij} = 0$, $j \geqslant i$, when A is in normal form. In this case all latent roots of A are 0.

~~Thus $y'x > 0$~~

and ~~by. 6.8~~, x ~~is associated with a classical submatrix of order 1.~~

~~Since there is only one linearly dependent latent vector, it follows that ρ is a single latent root.~~

8.9.

Inequalities for the elements of the latent vector x.

We may use the argument of 8.6. to give some new inequalities for the latent vector x associated with ρ, when $A \supset 0$ is irreducible.

If $x_1 \geq x_2 \geq \cdots \geq x_n > 0$,

we have from

$$\rho = \left(\sum_{j=1}^{n} a_{ij} x_j\right)/x_i$$

$$\frac{x_k}{x_{k+1}} \leq \frac{(n-k)\rho - t_2}{s_{21}} = \delta_k, \text{ say}$$

by an argument similar to that leading to (9).

Hence $$\frac{x_k}{x_l} \leq \delta_k \delta_{k+1} \cdots \cdots \delta_{l-1} = \Delta_{kl}, \text{ say}, \quad k < l \qquad (14)$$

and putting $\Delta_{kk} = 1$

(where x is normalised so that $\sum_{i=1}^{n} x_i = 1$),

$$1 = \sum_{i=1}^{n} x_i \leq \left[\sum_{i=1}^{k-1} \Delta_{ik} + (n - k + 1)\right] x_k ,$$

$$1 = \sum_{i=1}^{n} x_i \geq \left[k + \sum_{i=k+1}^{n} (1/\Delta_{ki})\right] x_k ,$$

whence

$$\frac{1}{\sum_{i=1}^{k-1} \Delta_{ik} + n - k + 1} \leq x_k \leq \frac{1}{k + \sum_{i=k+1}^{n} (1/\Delta_{ki})} . \quad (15)$$

In particular

$$1/n \leq x_1 \leq \frac{1}{\sum_{i=1}^{n} (1/\Delta_{1i})} = 1/L, \text{ say }, \quad (16)$$

$$\text{say } \frac{1}{K} = \frac{1}{\sum_{i=1}^{n} \Delta_{in}} \leq x_n \leq \frac{1}{n} \quad (17)$$

and $L = K/\Delta_{1n}$.

Now let $z > 0$ be a vector, and $R = R(z) = \max_i r_i(z)$.

Then by Collatz's theorem

$$\rho \leq R ,$$

and if $a = \min_i a_{ii}$ and $b = \min_i (a_{ij} z_i)/z_i$, for $a_{ij} > 0$

then $t_2 \geq (n - k) a$ and $s_{21} \geq b$,

as A is irreducible.

Hence

$$x_k / x_{k+1} \leq \sigma_k \leq (n - k)(R - a)/ b = (n - k) q , \quad (18)$$

where $q = (R - a)/ b$,

a quantity independent of k.

We have $q \geq 1/(n - k)$, as $(n - k)(R - a) \geq (n - k) R - t_2 \geq s_{21} \geq b$.

Hence $x_k / x_i \leq \Delta_{ki} \leq [(n - k)!/(n - i)!] \; q^{i-k}$,

and in particular

$$x_1/x_n \leq (n - 1)! \; q^{n-1} . \quad (19)$$

From (16) and (17) we have

$$L \geq \sum_{i=1}^{n} \frac{(n-i)!}{(n-1)\, q^{i-1}} = \sum_{i=0}^{n-1} \frac{(n-i-1)!}{(n-1)!\, q^{i}} \quad (20)$$

and $$K \leq \sum_{i=1}^{n} \frac{(n-i)!\, q^{n-i}}{1} = \sum_{i=0}^{n-1} i!\, q^{i}. \qquad (21)$$

~~and~~ Substituting in (16) and (17) we obtain

$$x_1 \leq 1/\left(\sum_{i=0}^{n-1} \frac{(n-i-1)!}{(n-1)!} \frac{1}{q^{i}} \right)$$

$$x_n \geq 1/\left(\sum_{i=0}^{n-1} i!\, q^{i} \right).$$

Of course much better bounds may be found in any particular case for t_2, δ_{21}, etc., by closer inspection of A, and hence (18), (19), (20), (21), may be improved.

When $A > 0$, $\delta_{21} \geq k(n-k)\, b$

and hence $$\delta_k \leq \frac{(R-a)}{k\, b} = \frac{q}{k}, \qquad (22)$$

and this yields

$$x_k/x_l \leq A_{kl} \leq [(k-1)!/(l-1)!]\; q^{l-k}, \; k < l.$$

In particular

$$x_1/x_n \leq q^{n-1}/(n-1)!,$$

$$K \leq \sum_{i=0}^{n-1} \frac{(n-i-1)!}{(n-1)!} q^{i},$$

$$L \geq \sum_{i=1}^{n} \frac{q^{i-1}}{(i-1)!} = \sum_{i=0}^{n-1} \frac{q^{i}}{i!}.$$

However when $A > 0$, these inequalities may be improved on by a slightly different method.

For $$\rho = \sum_{j=1}^{n} \frac{a_{lj} x_j}{x_l} = \sum_{j=1}^{k} \frac{a_{lj} x_j}{x_l} + \sum_{j=k+1}^{n} \frac{a_{lj} x_j}{x_l}.$$

$$\geq \frac{x_k}{x_l} \sum_{j=1}^{k} a_{lj} + \sum_{j=k+1}^{l} a_{lj}, \text{ when } l > k.$$

Suppose, as before,

$a = \min a_{ii}$ $\min (a_{ij} z_i)/z_i = b$, $R = \max r_i(z)$.

Since, $\rho \leq R$,

$$R \geq (x_k/x_l)\,kb + (l - k - 1)\,b + a\}\ ,$$

whence $x_k/x_l \leq (R - (l - k - 1)\,b - a)/kb = \epsilon_{kl}$, say,

$$(x_1/x_n) \leq (R - (n-2)\,b - a)/b = \epsilon_{1n}\ . \qquad (24)$$

~~and~~ If $\epsilon_{il} = 1$

we may replace A_{kl} by ϵ_{kl} in (15), (16), and (17).

Thus

$$1/\left(\sum_{i=1}^{k-1} \epsilon_{ik} + (n - k + 1)\right) \leq x_k \leq 1/\left(k + \sum_{i=k+1}^{n} (1/\epsilon_{ki})\right) \qquad (25)$$

and in particular

$$1/n \leq x_1 \leq 1/\sum_{i=1}^{n} (1/\epsilon_{1i})\ , \qquad (26)$$

$$1/\left(\sum_{i=1}^{n} \epsilon_{in}\right) \leq x_n \leq 1/n\ . \qquad (27)$$

That (24), is better than (23), and that (25), (26), (27) are better than ~~(17), and~~ the corresponding relations arising from (22), (which in turn are better than those derived from (18)) follow from

$$\epsilon_{kl} \leq (R - q)/kb = q/k\ .$$

If $l > k + 1$ the bound for x_n/x_l derived from (22) contains $q^{(l-k)}$, $l - k > 1$, as a factor and hence the improvement may be considerable.

8.10.

The inequalities of 8.9. may be applied to improve the bounds for ρ, given by Collatz's theorem.

Let $A > 0$, be irreducible, and let

$$\underline{t} = \underline{t}\,(e) = e'\,A\ ,$$

$$t_m = \max_i\ t_i = T\ ,$$

$$t_m = \min_i t_i = t \, .$$

From Collatz's theorem applied to A' we have,

$$\text{if } T > t, \qquad t < \rho < T \, .$$

Suppose x is the latent column vector associated with ρ, then $x > 0$. Suppose that $\sum_{i=1}^{n} x_i = 1$, and that we have obtained by one of the methods of 8.8. a positive lower bound for $x_n = \min x_i$. ~~and an upper bound for $x_1 = \max x_i$~~

Assume $x_n \geq \alpha > 0$, $\cancel{x_1 \leq \beta}$

As
$$\rho x = A x \, ,$$
$$\rho x_i = \sum_{j=1}^{n} a_{ij} x_j \, , \qquad i = 1, \ldots, n \, ,$$

whence

$$\rho \sum_{i=1}^{n} x_i = \sum_{i,j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} T_j x_j \leq T \sum_{i=1}^{n} x_i - (T - T_m) x_m \, ,$$

whence $\rho \leq T - \alpha (T - t)$.

Similarly $\rho \geq t \sum_{i=1}^{n} x_i + (T_M - t) x_M \geq t + \alpha (T - t)$,

Similar relations may be found if the lower bound for $\min y_i$ is known, where y_i is the latent row vector associated with ρ.

Suppose $\min y_i > \beta$,

When $\sum_{i=1}^{n} y_i = 1$

and $\underline{r} = r(e)$, $R = \max r_i$, $r = \min r_i$.

We obtain

$$r + \beta (R - r) \leq \rho \leq R - \beta (R - r) \, .$$

8.11.

In 8.9 we obtained an upper bound for x_1/x_n, when x is the latent column vector associated with ρ, in an irreducible weakly positive A. For $A > 0$ Ledermann (1950,a) derived an upper bound for x_n/x_1. We shall take this opportunity of improving this bound, using essentially the same method as Ledermann. The generalisation of this bound to irreducible $A \geq 0$ is trivial.

Let
$$r_M = \max_i r_i(e) = R,$$
$$r_m = \min_i r_i(e) = r,$$
and $x_1 \geq x_2 \geq \ldots \geq x_n > 0$.

We have $\rho x_i = \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, \ldots, n,$

and so
$$\frac{r_i x_n}{x_i} \leq \rho \leq \frac{r_i x_1}{x_i}, \quad i = 1, \ldots, n. \tag{28}$$

If $R > r$,

then $x_1 > x_n$, by 8.1.

For if $x_1 = x_n$, $x = e$, whence $A e = \rho e$, and $r = \rho = R$.

If further, $A > 0$, (28) may be improved to
$$(r_i x_n)/x_i < \rho < (r_i x_1)/x_i, \tag{29}$$

as then $r_i x_n = x_n \sum_{j=1}^{n} a_{ij} < \sum_{j=1}^{n} a_{ij} x_j = \rho x_i$, etc.

If in (29) we put $i = n$ on the left hand side, $i = 1$ on the right hand side we obtain
$$r_n < \rho < r_1,$$

whence

$$(r_n/r_1) < \max_{(r_i/r_j)<1} (r_i/ r_j) = d \quad , \quad \text{say} .$$

If in (29) we put $i = 1$ on the left hand side and $i = n$ on the right hand side we obtain

$$(r_1\, x_n)/ x_1 < \rho < (r_n\, x_1)/ x_n .$$

Hence

$$(x_n/ x_1) < (r_n/ r_1)^{1/2} < d^{1/2} < 1 . \tag{30}$$

This is Ledermann's inequality for $A > 0$.

However, in (28) we may put $i = M$ on the left hand side, $i = m$ on the right hand side and obtain for irreducible $A \supset 0$

$$(R\, x_n)/ x_M \leq (r_M x_n)/x_M \leq \rho \leq (r_m x_1)/x_m \leq (r\, x_1)/x_n ,$$

whence $(x_n / x_1) \leq (r/R)^{1/2}$. (31)

Evidently $(r/R) \leq d$

and $(r/R) = d$,

if and only if, either $r_i = r$ or $r_i = R$, for $i = 1, \ldots, n$.

Hence (31) is better than (29) except in that very special case.

8.12.

In this section we shall assume $x_1 > x_n$. This implies $n \geq 2$.

The equality is possible in (31) if $x_1 > x_n$, e.g.

$$A = \begin{bmatrix} \cdot & \cdot & 4 \\ 1 & \cdot & \cdot \\ \cdot & 1 & 1 \end{bmatrix} \quad \begin{array}{l} \rho = 2 , \quad x = \{2, 1, 1\} \\ x_1/x_3 = (r/R)^{1/2} = 1/2 \ ; \end{array}$$

A is irreducible and mod A = 1 .

It does not appear possible to give necessary and sufficient conditions depending on the position of zero elements only, for the equality to hold in (31).
To find a sufficient condition, we shall derive (31) in a different way.

We may put (Cf. § 13 , below)

$$A = X P X^{-1} , \qquad e = \{1, 1, \ldots . 1\} ,$$

where $X = \text{diag } (x_1 \ldots . x_n)$, $(x_i > 0)$
and $X e = x$, the latent column vector .
Then $Pe = \rho e$.

Now $$r_i/\rho = (x_i/\rho)\left\{\sum_{j=1}^{n}(p_{ij}/x_j)\right\} \leq (1/\rho)\ (x_1/x_n)\left(\sum_{j=1}^{n} p_{ij}\right)$$
$$= (x_1/x_n) , \tag{32}$$

and we deduce from (32) that

$$(r_k/\rho) = (x_1/x_n) \tag{33}$$

if and only if

$$x_k = x_1 \quad , \quad \text{and} \tag{34}$$

$$a_{kj} = p_{kj} = 0 \quad \text{unless} \quad x_j = x_n . \tag{35}$$

Similarly $$(r_i/\rho) \geqslant (x_n/x_1) \tag{36}$$

and $\quad (r_\ell/\rho) \quad = \quad (x_n/x_1) \qquad\qquad (37)$

if and only if

$$x_\ell \quad = \quad x_n \qquad\qquad (38)$$

and $\quad a_{ij} = \rho_{ij} = 0 \,, \quad$ unless $\quad x_j = x_1 \,. \qquad (39)$

Hence from (32) and (36)

$$(x_n/x_1) \;\leq\; (r_\ell/r_k)^{1/2}$$

for all $k, \ell, \qquad 1 \leq k, \ell \leq n$.

Putting, as we may, $r_k = R$, $r_\ell = r$, we obtain

$$(x_n/x_1) \;\leq\; (r/R)^{1/2} \;. \qquad\qquad (31)$$

We deduce that

$$(x_n/x_1) \;=\; (r_\ell/r_k)^{1/2} \qquad\qquad (40)$$

if and only if (34), (35), (38) and (39) are satisfied.
If (40) holds, (33) and (37) must hold and hence from
$(r_\ell/r_k)^{1/2} \geq (r/R)^{1/2} \geq (x_n/x_1)$ we see that

$$r_k = R \;, \quad r_\ell = r \;,$$

in (40).

Conversely if

$$(x_n/x_1) \;=\; (r/R)^{1/2}$$

it is obvious that (40) is satisfied for some k and ℓ .
Hence (31) holds if and only if (34), (35), (38) and (39)
are satisfied for some k and ℓ .

We have given necessary and sufficient conditions for (40)
in terms of the latent vector x and zeros amongst the a_{ki} .
One might suspect that it would be possible to replace x_i by
r_i, x_1 by R, x_n by r in (34), (35), (38) and (39).

(The reason being that if $A = X P X^{-1}$, for fixed P, $r_k = \sum_{j=1}^{n} a_{kj}$ will not decrease, r_i $(i \neq k)$ will not increase, when x_k is increased and the x_i $(i \neq k)$ kept constant).

That this is not so may be seen from an example.

$$A = \begin{bmatrix} . & 16 & . \\ . & . & 16 \\ 1 & 2 & . \end{bmatrix} \qquad \begin{array}{l} \rho = 8 , \\ x = \{ 4 \quad 2 \quad 1 \} ; \end{array}$$

$r_1 = r_2 = R = 16$, $\quad r_3 = r = 3$,

$a_{2j} = 0$ unless $j = 3$,

$a_{3j} = 0$ unless $j = 1, 2$.

(These are the analogues of (35), (38)).

$(x_m/x_1) = 1/4$,

but $(r/R)^{1/2} = \sqrt{3}/4$.

The relations (35) and (39) cannot simultaneously hold for all k, l such that $x_k = x_1$, $x_l = x_n$, unless there is an α such that $x_1 = x_\alpha > x_{\alpha+1} = x_n$. For let $x_1 = x_2 = x_\alpha > x_{\alpha+1}$, $x_{\beta-1} > x_\beta = x_{\beta+1} \ldots\ldots x_n$. Then $\alpha < \beta$, and from (35), (39), $M = 0$, where $M : [a_{ij}]$, $1 \leq i \leq \alpha$, and $\beta \leq i \leq n$, $\alpha < j < \beta$. + 1 .

Hence A is reducible when $\beta > \alpha + 1$ and the result follows.

We deduce that if (40) holds either $x_1 = x_2$ or $x_{n-1} = x_n$ if $n > 2$.

This is obvious if $x_1 = x_2 = x_\alpha > x_{\alpha+1} = x_{\alpha+2} = = x_n$, While if there is some x_i , $x_1 > x_i > x_n$, we must have either (35) for some $x_k = x_1$ and an $x_\rho = x_1$ such that (35) does not hold for x_ρ, or (39) for some $x_l = x_n$, and an $x_\sigma = x_n$, such than (39) does not hold for x_σ .

If (34) and (35) hold, $a_{kk} = 0$.

If (38) and (39) hold, $a_{ll} = 0$.

Hence if either $a_{ii} = 0$ for all $r_i = R$, or if $a_{ii} = 0$ for all $r_i = r$, (40) cannot be satisfied, and hence

$$(x_n/x_1) < (r/R)^{1/2} . \qquad (41)$$

In other words it is a necessary condition for (40) that there should be integers k, l , for which $r_k = R$, $r_l = r$ and $a_{kk} = a_{ll} = 0$.

In particular, if $A > 0$, (40) cannot be satisfied and (41) must hold.

8.13.

We shall call a matrix $P \supset 0$ a "P - matrix" if $P e = \rho e$. (i.e. $\sum_{j=1}^{n} p_{ij} = \rho$, $i = 1, \ldots, n$) .

$\underline{A} \supset 0$, will be called a "P' - matrix" if A' is a P - matrix. (i.e. $\sum_{i=1}^{n} a_{ij} = \rho$, $j = 1, \ldots n$).

When $\rho = 1$, the P - matrix is called a "stochastic" or "probability" matrix. Such matrices occur in the theory of Markoff chains.

If $x = \{x_1 \ldots x_n\}$ we shall denote by $X = \text{diag } x$, the diagonal matrix $X = \text{diag}\ (x_1 \dots x_n)$

In 8.12 we used the fact that every irreducible $A \supset 0$ is similar to a P - matrix on transformation by a diagonal matrix with positive diagonal elements. We shall now enunciate this more precisely and supply a proof.

Theorem $A \supset 0$ is irreducible, $x > 0$ the latent column vector, $y' > 0$ the latent row vector associated with the positive greatest latent root.

The matrix $X = \text{diag}\ x$, is the unique diagonal matrix with positive elements such that $X^{-1}AX$ is a P-matrix.

The matrix $Y = \text{diag}\ y$ is the unique diagonal matrix with positive diagonal elements such that YAY^{-1} is a P′-matrix.

Let $X = \text{diag}\ x$. Then $Xe = x$.
Let $h = X^{-1}AXe = X^{-1}Ax = \rho X^{-1}x = \rho e$.

Thus $X^{-1}AX$ is a P - matrix.

Let $Z = \text{diag}\ z$, $z > 0$, and let $Z^{-1}AZ$ be a P - matrix,

$$Pe = \sigma e.$$

Hence σ is the positive greatest latent root of P, and hence of A. Thus $\sigma = \rho$,

and $Z^{-1}AZe = \rho e$,

whence $Az = \rho Ze = \rho z$.

It follows that $z > 0$ is the (unique) latent column vector associated with ρ.

Hence $z = x$, and $Z = X$.

The proof of the second part of the theorem is similar.

CHAPTER 9.

PART I - INTRODUCTION

9.1.

In this chapter we shall be concerned with square n x n matrices P : $[p_{ij}]$, such that $\sum_{j=1}^{n} p_{ij} = 1$, i = 1, ..., n (now called P - matrices, and more generally matrices Q : $[p_{ij}]$ such that $\sum_{j=1}^{n} p_{ij} \leq 1$. These latter will be called Q - matrices. The theory of P - matrices assumes great importance in the theory of Markoff Chains C.f. (Frechet (1938), V. Romanovsky (1936)). Various methods have been used to study (the methods of) P - matrices. Some related matrices were recently considered by W. Ledermann, (1950, b) by a method similar to that of Frobenius. In this chapter we shall use a method of V. Romanovsky (1936), though when this was first written we were unaware of the work of this author. There is some justification for this chapter, in spite of Romanovsky's paper, as that author frequently appealed to Frobenius' theorems, proved by determinantal considerations. The results of this chapter are entirely independent of such considerations (except 9.50). Only a few of the results of this chapter are new.

If A is a square n x n matrix and $A \supset 0$, there is clearly an $m > 0$ such that A / m is a Q - matrix. The matrix Q has the same latent vectors as A, and the latent roots of Q are those of A divided by m. Hence the restriction of Q - matrices is slight.

We shall prove in this chapter, several results proved before and but for the use of the results of Chapters 3 & 5, this chapter is complete in itself, with one or two exceptions.

In this chapter E will denote the set 1, 2, ... n, as usual.

PART 2. - General Properties of Q - Matrices.

9.2.

Let y' be a latent row vector of Q associated with the latent root λ.

Then $$\lambda y' = y' Q ,$$

or $$\lambda y_j = \sum_{i \in E} y_i p_{ij} , \qquad j \in E .$$

Hence $$|\lambda| |y_j| \leq \sum_{i \in E} |y_i| p_{ij} , \quad j \in E ,$$

and so $$|\lambda| \sum_{j \in E} |y_j| \leq \sum_{i, j \in E} |y_i| p_{ij} \leq \sum_{i \in E} |y_i| .$$

It follows that $$|\lambda| \leq 1 ,$$

as $$\sum_{i \in E} |y_i| > 0 , \qquad y' \text{ being non-zero.}$$

9.3.

In the remaining sections we shall assume that $|\lambda| = 1$.

By 2 , $$|y_j| \leq \sum_{i \in E} |y_i| p_{ij} ,$$

whence $$\sum_{j \in E} |y_j| \leq \sum_{i \in E} |y_i| ;$$

but as $$\sum_{j \in E} |y_j| = \sum_{i \in E} |y_i| ,$$

we must have $$|y_j| = \sum_{i \in E} |y_i| p_{ij} .$$

Hence if $$y_j = 0,$$

then by inspection of 2, either $p_{ij} = 0$ or $y_i = 0$, $i \in E$.

Also $\sum_{j \in E} p_{ij} = 1$ if $y_i \neq 0$.

9.4.

With the assumptions for λ, y', as above

$$\lambda y_j = \sum_{i \in E} (y_i \; p_{ij}),$$

and so $\arg \lambda + \arg y_j = \arg (\sum_{i \in E} y_i \, p_{ij}), \; j \in E$,

while by 3 $|y_j| = \sum_{i \in E} |y_i| \, p_{ij}, \quad j \in E$.

It follows that $y_j \neq 0$ and $p_{ij} > 0$

implies either $\arg y_i = \arg y_j + \arg \lambda$,

or $y_i = 0$,

as all non-zero $p_{ij} \, y_i$ have equal arguments.

Sections 3 and 4 contain the fundamental relations which will be used as means for the investigation of latent roots and vectors of Q - matrices associated with latent roots of unit modulus.

PART 3 - Properties of Irreducible P - Matrices.

9.5.

Let P be irreducible and $y' P = \lambda y'$, $|\lambda| = 1$.

Then $|y_i| > 0$, all $i \in E$.

Let $i \in E_1$ when $|y_i| > 0$.

Let $i \in E_2$ when $y_i = 0$.

Clearly $E_1 \cup E_2 = E$,

and as some $y_i \neq 0$, E_1 is non-empty.

Let $j \in E_2$ and $i \in E_1$.

By 3, $p_{ij} = 0$. ~~where $P : p_{ij}$~~

Hence $P_{12} = 0$, where $P_{12} : [p_{ij}]$ $(i \in E_1, j \in E_2)$,

and if $E_1 \neq 0$, P is reducible.

Hence E_2 is empty, and the result follows.

9.6.

If $\lambda, |\lambda| = 1$ is a latent root of an irreducible P, there is only one (linearly independent) latent row vector associated with it.

Suppose y', w' are latent row vectors associated with λ.

By 5.
$$|y_i| > 0 \qquad |w_i| > 0, \qquad i \in E.$$
Let
$$v = y_1 w - w_1 y.$$
Then
$$v'P = \lambda v'$$
and
$$v_1 = y_1 w_1 - w_1 y_1 = 0.$$

But by 5, this implies that $v = 0$, and hence the result follows .

9.7.

Let us define mod P as in 5.5. ; i.e.

Let mod P = r be the maximum number of sets E_α such that

(1) $\bigcup_{i \in E_\alpha} E_\alpha = E$,

(2) $E_\alpha \cap E_\beta = 0$,

and, (3), $E_\alpha = E_\beta$ when $\beta \equiv \alpha$, mod r.

Also, (4), $P_{\alpha\beta} = 0$ unless $\beta \equiv \alpha + 1$, mod r,

where $P_{\alpha\beta} : [p_{ij}]$, $i \in E_\alpha$ $j \in E_\beta$.

We have shown that $1 \leq r \leq n$.

The sets E_α $\alpha = 1, \ldots r$ will be called "The modular sets". They are unique, except that we may put $\alpha + \gamma$ for α, $\alpha = 1, \ldots r$, ~~apart from an additive constant~~, by 5.6., Corollary to ~~the~~ theorem 7.

9.8.

Let P be ~~simple~~ irreducible and let $i \in E_\alpha$, a modular set. Then $\sum_{j \in E_{\alpha+1}} p_{ij} = 1$, $\alpha = 1, \ldots, r$.

Let $i \in E_\alpha$.

$$1 = \sum_{j \in E} p_{ij} = \sum_{j \in E_{\alpha+1}} p_{ij},$$

as $p_{ij} = 0$, when $i \in E_\alpha$, $j \notin E_{\alpha+1}$.

9.9.

Let us denote by L_r the set of the r r-th roots of unity. If $\lambda \in L_r$, then λ is a latent root of any irreducible P of mod r. It has a unique associated latent vector x, such that

$$x_i = \lambda^{\alpha}, \quad \text{when} \quad i \in E_{\alpha}, \quad \alpha = 1, \dots, r,$$

where the E_{α} are the modular sets.

Let $i \in E_{\alpha}$, and let x be defined as above. Then by 9.8.

$$\sum_{j \in E_*} p_{ij} x_j = \sum_{j \in E_{\alpha+1}} p_{ij} x_j = \lambda^{\alpha+1}, \; (i \in E_{\alpha},)$$

~~Hence~~

$$\cancel{\sum_{j \in E_{\alpha}} p_{ij} x_j} = \lambda^{\alpha} . \lambda = \lambda . x_i .$$

It should be noted that in the case $i \in E_r$ this holds precisely because $\lambda^r = 1$.

We have proved that λ is a latent root and that it has a latent column vector of the required form associated with it. But, by 6, there is a single latent row vector associated with λ. Hence x is the unique latent column vector associated with λ.

9.10.

Every irreducible P contains a set L_r of latent roots, for some $r \geq 1$. In particular 1 is a latent root of every such matrix.

For $\text{mod } P = r \geq 1$, and $1 \in L_r$, $r \geq 1$.

The result now follows by the first part of 9.

9.11.

If λ , $|\lambda|$ = 1, ~~[illegible]~~ is a latent root of an irreducible P, there is an integer s, $1 \leq s \leq n$, such that $\lambda \in L_s$ (i.e. $\lambda^s = 1$).

Let y' be a latent row vector associated with λ. We may arbitrarily choose some $\arg y_i = 0$ (As $|y_i| > 0$ by 5).

Let $i \in F_0$ where $\arg y_i \equiv 0$, mod 2π.

Every column of an irreducible matrix contains at least one non-zero element.

Let $p_{ij} > 0$ for $j \in F_0$.

By 4, $\arg y_i = \arg \lambda$ as $y_i \neq 0$, by 5.

Let $i \in F_{-1}$ where $\arg y_i \equiv \arg \lambda$, (mod 2π).

Then F_{-1} is non-empty.

We have constructed non-empty sets F_0, F_{-1}. Continuing this we may similarly construct non-~~null~~ empty sets $F_{-\alpha}$, $\alpha = 1, 2, 3, \ldots$, such that $i \in F_{-\alpha}$ when $\arg y_i \equiv \alpha \arg \lambda$, mod 2π.

Let $i_\alpha \in F_{-\alpha}$, $\alpha = 1, 2, 3, \ldots,$.

But $i_\alpha \in E$, and so there are no more than n distinct i_α.

Hence there is a k', $2 \leq k' \leq n$, such that there is a j', $1 \leq j' < k'$, for which $i_{k'} = i_{j'}$.

If k is the smallest such k', and j the largest j' for k,

$$k \arg \lambda \equiv j \arg \lambda \quad \text{mod } 2\pi,$$

and putting $s = k - j$

we have $\lambda^s = 1$,

and $\lambda^t \neq 1$, $1 \leq t < s$, if $s > 1$.

Note As $\alpha \arg \lambda \neq \beta \arg \lambda$, mod 2π

when $1 \leq \alpha \leq s$, and $0 \leq \beta < \alpha$,

and $s \arg \lambda = 0$ mod 2π,

~~hence~~ it follows that $j = 0$, and $k = s$.

9.12.

If $F_{-\alpha}$, $\alpha = 0, 1, 2, \ldots, s-1$, is defined as in 11, then

$$F_{-\alpha} = F_{-\beta}, \text{ when } \beta \equiv \alpha \mod s.$$

If F_{α}, for real integers α, is defined as

$$F_{\alpha} = F_{-\beta}, \beta \geq 0, \text{ when } \alpha = -\beta, \mod s,$$

then $i \in F_{\alpha}$ when, $\arg y_i \equiv -\alpha \arg \lambda$, mod 2π.

Let $\beta \equiv \alpha$, mod s, $0 \leq \alpha, \beta \leq s - 1$.

When $i \in F_{-\alpha}$, $\arg y_i \equiv \alpha \arg \lambda \equiv \beta \arg \lambda$, mod 2π.
Hence $y_i \in F_{-\beta}$.

~~As $(\beta - \alpha) \arg \lambda = p s \arg \lambda \equiv 0$ mod 2π~~
~~where p is some integer.~~

Hence $F_{-\alpha} \prec F_{-\beta}$, and reversing α, β in the above we obtain $F_{-\beta} \prec F_{-\alpha}$.

Thus $F_{-\alpha} = F_{-\beta}$.

Let $i \in F_{\alpha}$ for some real integer α, and let $\beta > 0$ be an integer such that

$$\alpha \equiv -\beta \mod s.$$

Then $\arg y_i \equiv \beta \arg \lambda = -\alpha \arg \lambda$, mod 2π.

Conversely if $\arg y_i = -\alpha \arg \lambda$, mod 2π,

then $\arg y_i = \beta \arg \lambda$, mod 2π;

whence $i \in F_{-\beta}$

and by $F_{\alpha} = F_{-\beta}$, $i \in F_{\alpha}$.

9.13.

If λ, $\lambda \in L_s$ is a latent root of an irreducible P, and F_α, $\alpha = 1, \ldots, s$ is defined as in 12, then $F_\alpha \cap F_\beta = 0$, $1 \leq \alpha, \beta \leq s$,

$$\bigcup_{\alpha=1}^{s} F_\alpha = E .$$

Let $i \in F_\alpha$.

Then $\arg y_i \equiv -\alpha \arg \lambda \not\equiv -\beta \arg \lambda$, mod 2π,

as $|\beta - \alpha| < s$. ~~mod~~

Hence $i \notin F_\beta$, and so $F_\alpha \cap F_\beta = 0$.

Let $\bigcup_{\alpha=1}^{s} F_\alpha = F$, and $E - F = F'$. Suppose F' is non-empty.

Let $i \in F$. There is an α, $1 \leq \alpha \leq s$, such that $i \in F_\alpha$.

Thus $\arg y_i = -\alpha \arg \lambda$.

Let $j \in F'$. Then $\arg y_i \not\equiv -\alpha \arg \lambda$, mod 2π, $1 \leq \alpha \leq s$.

Hence, by 4, $p_{ij} = 0$, $i \in F$, $j \in F'$.

By definition P is reducible.

Hence $F' = 0$, and $F = E$.

9.14.

If P is irreducible, and the F_α are defined as in 9.12, then $p_{ij} \neq 0$ implies $i \in F_\alpha$, $j \in F_{\alpha+1}$, ~~be~~ for some α.

By 13, there is an α such that $j \in F_{\alpha+1}$.

Let $j \in F_{\alpha+1}$.

Then $\arg y_j \equiv -(\alpha+1) \arg \lambda$, mod 2π, As $p_{ij} > 0$, by 4,

$$\arg y_i = -\xi \alpha \xi \arg \lambda ,$$

Whence $i \in F_\gamma$.

9.15.

If λ is a latent root of P, and s is the smallest integer such that $\lambda^s = 1$, s divides r, where r = mod A. For the sets F_α, $\alpha = 1, \ldots, s$ satisfy the conditions of 5.5, Hence 5.6, Theorem 7, holds.

9.16.

The set L_r contains all the latent roots of unit modulus of an irreducible P if and only if mod P = r.

Let mod $P = r$.

By 9, the set L_r are latent roots of P, of unit modulus. If λ, $|\lambda| = 1$, is a latent root of P, there is a t such that $\lambda^t = 1$; by 11. If s is the smallest such t, then s divides r.

Hence $\lambda^r = 1$; and $\lambda \in L_r$. Hence if mod P = r , the set L_r are all the latent roots of unit modulus of P.

Now let L_r be all the latent roots of unit modulus . Let mod P = k .

By the above result we immediately have k = r .

This completes the proof.

9.17.

Let mod P = r . There is a single latent row vector

y' associated with λ, $\lambda \in L_r$. If $\lambda^s = 1$, $\lambda^t \neq 1$, $1 \leq t < s$, and the sets F_α are defined as in 9.12, then y' is of the form

$$y_i = v_i \lambda^{r-\alpha}, \quad v_i > 0,$$

where $i \in F_\alpha$.

By 9.6, there is a single y' associated with λ.

By 5, $|y_i| = v_i > 0$.

As $\bigcup^s F_\alpha = E$, there is an α such that $i \in F_\alpha$.

As in 11 $\arg y_i \equiv - \alpha \arg \lambda$, mod 2π.

Hence $y_i = v_i \lambda^{r-\alpha}$ as $\lambda^r = 1$, by 15.

As $\lambda^s = 1$ and $F_{\alpha+s} = F_\alpha$, this result is true even for $\alpha > s$, or $\alpha \leq 0$.

9.18.

Let mod P = r, and E_α, $\alpha = 1, \ldots, r$ be the modular set. The latent row vector y' associated with λ satisfies

$$y_i = v_i \lambda^{r-\alpha}, \quad v_i > 0,$$

where $i \in E_\alpha$.

Let $\lambda^s = 1$, $\lambda^t \neq 1$, $1 \leq t < s$.

By 5.6, Theorem 7, there is an h such that $r = hs$ and

$$F_\beta = E_\beta \cup E_{\beta+s} \cup \cdots \cup E_{\beta+(h-1)s}$$
$$\beta = 1, \ldots, s,$$

when the F_α, the sets of 9.12, are properly numbered.

Hence $E_\alpha \subset F_\alpha$

as $F_\alpha = F_{\alpha - ks}$, k some integer.

Let $i \in E_\alpha$. Then $i \in F_\alpha$.

Then by 9.17., we have

$$y_i = v_i \lambda^{r-\alpha}, \qquad v_i > 0 .$$

9.19.

Let mod P = r, and y' be the latent row vector associated with $\lambda \in L_r$.

Then $|y_i| = v_i$ does not depend on λ.

Let λ be a primitive r-th root of unity, viz.

$$\lambda^r = 1, \quad \lambda^t \neq 1, \quad 1 \leq t < r.$$

Let $i \in E_\alpha$, and let y' be associated with λ.

By 9.18, $y_i = v_i \lambda^{r-\alpha}$.

If $j \in E_{\alpha+1}$, $v_j = \sum_{i \in E} v_i p_{ij} = \sum_{i \in E_\alpha} v_i p_{ij}$

as $p_{ij} = 0$, $i \in E_\alpha$, $j \notin E_{\alpha+1}$.

Let $1 \leq m \leq r$. Then $\lambda^m \in L_r$.

Let w' be a row vector such that $w_i = v_i \lambda^{m(r-\alpha)}$ when $i \in E_\alpha$.

Let $j \in E_{\alpha+1}$.

Then $\sum_{i \in E} w_i p_{ij} = \sum_{i \in E_\alpha} w_i p_{ij} = \sum_{i \in E_\alpha} v_i \lambda^{m(r-\alpha)} p_{ij} = v_j \lambda^{m(r-\alpha)}$ by letting $\lambda = 1$, in 18,

and so $\sum_{i \in E} w_i p_{ij} = \lambda^m v_j \lambda^{m(r-\alpha-1)} = \lambda^m w_j$.

Hence w' is a latent row vector associated with λ^m. As there is an μ such that $\lambda^m = \mu$ for every $\mu \in L_r$, and as, by 9, there is only one linearly independent latent row vector associated with $\mu \in L_r$, the result follows.

9.20

If mod $P = r$, and v_i is defined as in 17, then $\sum_{i \in E_\alpha} v_i = 1/r$, $\alpha = 1, \ldots, r$, when y' is normalized so that $\sum_{i \in E} v_i = 1$.

Let $j \in E_{\alpha+1}$.

By definition of E_α,

$$\sum_{i \in E} v_i \, p_{ij} = \sum_{i \in E_\alpha} v_i \, p_{ij},$$

and by 8, $$\sum_{j \in E_{\alpha+1}} v_j = \sum_{j \in E_{\alpha+1}} \sum_{i \in E_\alpha} v_i p_{ij} = \sum_{i \in E_\alpha} v_i .$$

Hence $$\sum_{i \in E_1} v_i = \sum_{i \in E_2} v_i = \qquad = \sum_{i \in E_r} v_i = 1/r.$$

9.21.

In the case of latent column vectors 19, has this analogue : we have $|x_i| = 1$, $i \in E_\alpha$, where x is the latent column vector associated with any $\lambda \in L_\mu$.

The result corresponding to 20, is this : we have $\sum_{i \in E_\alpha} |x_i| = n_\alpha$ $\alpha = 1, \ldots, r$, for any $\lambda \in L_r$, where n_α is the number of members of E_α and $|x_i| = 1$.

9.22.

If $\lambda \in L_r$, there is a single classical canonical submatrix associated with λ , and it is of order 1.

Let p be the number of classical canonical submatrices associated with λ . By 6.3, the number of linearly independent latent column vectors equals p. But there is only one such vector, by 9. Hence p = 1 .

Let x be the latent column vector, y' be the latent row vector associated with λ .

By 9 and 17, $y'x = \sum_{i = E_\alpha} v_i \lambda^{r-\alpha} \lambda^{\alpha} = \sum_{i \in E_\alpha} v_i = 1/r\theta$,

~~as v_i 0, i E~~ . by 20.

Hence by 6.4, Theorem 2, there is a classical canonical submatrix of order 1 associated with λ .

Combining these two statements the result follows.

9.23.

Let mod P = r . Then $\lambda \in L_r$ is a single latent root, (its multiplicity is 1).

By 6.3, Theorem 1, the multiplicity of a latent root equals the number of latent column vectors associated with it, if all classical canonical submatrices associated with it are of order 1. Hence by 22, λ is single.

9.24.

Summary of Part 3.

Let P be irreducible. The set of all latent roots of unit modulus of P is L_r, if and only if mod P = r. Each $\lambda \in L_r$ is simple and the classical canonical submatrix associated with λ is of order 1. Associated with λ there is a unique latent column vector x and a unique latent row vector y', such that

$$x_i = \lambda^{\alpha}, \text{ when } i \in E_{\alpha},$$

$$y_i = v_i \lambda^{r-\alpha} \text{ when } i \in E_{\alpha},$$

where the $E_{\alpha} = 1, \ldots, r$ are the modular sets and $v_i, i = 1, \ldots, r$ is independent of λ. The v_i satisfy

$$\sum_{i \in E_1} v_i = \sum_{i \in E_2} v_i = \sum_{i \in E_r} v_i .$$

PART 4. Some Properties of Q Matrices.

9.25.

Definition. We shall say that the Q - matrix Q is of class p, (class Q = p) when it contains p irreducible principal P - submatrices.

(A principal submatrix $P \prec Q$ is such that

$P : [p_{ij}]$, $i, j \in G \prec E$. Cf. 6.1.)

9.26.

Let P be an irreducible P matrix. Then class $P = 1$.

Let $P_1 \prec P$, be an irreducible principal P - submatrix, say

$$P_1 : \left[p_{ij}\right], \quad i, j \in G \prec E.$$

Let $i \in G$; then $\sum_{j \in G} p_{ij} = 1$.

But $\sum_{j \in E} p_{ij} = 1$,

and so, as $p_{ij} \geq 0$, it follows that

$$p_{ij} = 0, \quad i \in G, \quad j \in E - G.$$

Hence if $E - G$ is non-~~null~~ empty, P is ~~ir~~reducible. Thus $G = E$, and $P_1 = P$.

There is only one irreducible principal P - matrix of P.

9.27.

Let P be a P - matrix.

Then class $P \geq 1$.

Let class $P = p$.

By 3.2, Theorem 1, there are sets E_α, $\alpha = 1, 2, \ldots, k$, such that

$$p_{ij} = 0 \quad \text{if} \quad i \in E_1, \quad j \notin E_1, \text{ i.e. } j \in E - E_1,$$

and $P_1 : \left[p_{ij}\right]$, $i, j \in E_1$, is irreducible.

Let $i \in E_1$. Then

$$1 = \sum_{j \in E} p_{ij} = \sum_{j \in E_1} p_{ij}.$$

The matrix P_1 is thus an irreducible P - matrix. Hence $p \geq 1$.

9.28.

Let P_1 be a principal P - submatrix of Q, and $P_1 \neq Q$.

If $p_{ii} \in P_1$, $p_{ij} \notin P_1$, then $p_{ij} = 0$.

The proof is similar to that of 9.26.

9.29.

Let class $P = p$, and let $P_\alpha : [p_{ij}]$, $i, j \in E_\alpha$, $\alpha = 1, \ldots, p$, be the irreducible principal P - submatrices of Q.

If we define $\sum_{\alpha=1}^{p} E_\alpha = G$, $F = E - G$, and

$Q_1 : [p_{ij}]$, $i, j \in F$, then class $Q_1 = 0$.

For if $class(Q_1) > 0$, there is a $P_{\rho+1} \prec Q$, where $P_{\rho+1}$ is an irreducible P - matrix.

Hence there is a set $E_{\rho+1} \prec F$ such that

$P_{\rho+1} : [p_{ij}]$, $i, j \in E_{\rho+1}$.

But $E_{\rho+1} \cap E_\alpha = 0$, $\alpha = 1, 2, \ldots, p$,

as $G \cap F = 0$.

Hence $P_{\rho+1} \neq P_\alpha$, $\alpha = 1, \ldots, p$,

and thus class $Q \geq p + 1$.

Hence class $Q_1 = 0$.

We shall always give this meaning to G and F.

We have $E = G \cup F$, $G \cap F = 0$, and of course one of G and F may be empty.

9.30.

The set E_α of 29 is a set of the decomposition of 3.2.

Let us denote the sets of 3.2 by $\overline{E}_\beta$, $\beta = 1, \ldots, k$, and let $P_{\beta\beta} = [p_{ij}]$, $i, j \in \overline{E}_\beta$.
There is a β such that $E_\alpha \cap \overline{E}_\beta \neq 0$. Let γ be the smallest such β.
If $E_\alpha - (E_\alpha \cap \overline{E}_\gamma) \neq 0$,
then $p_{ij} = 0$, $i \in (E_\alpha \cap \overline{E}_\gamma)$, $j \in E_\alpha - (E_\alpha \cap \overline{E}_\gamma)$, by 3.2, Theorem 1, (ii), as $j \in \overline{E}_\beta$ with $\beta > \gamma$.
Hence $P_{\gamma\gamma}$ is reducible contrary to 3.2, Theorem 1 (i).
It follows that $E_\alpha - (E_\alpha \cap \overline{E}_\gamma) = 0$, or $\overline{E}_\gamma \supset E_\alpha$.

If $\overline{E}_\gamma - (E_\alpha \cap \overline{E}_\gamma) \neq 0$,

then $p_{ij} = 0$, $i \in (E_\alpha \cap \overline{E}_\gamma)$, $j \in \overline{E}_\gamma - (E_\alpha \cap \overline{E}_\gamma)$, by 28, as $p_{ii} \in P_\alpha$, $p_{ij} \notin P_\alpha$.

Hence P_α is reducible, contrary to assumption.

Thus $E_\alpha \supset \overline{E}_\gamma$,

and so $E_\alpha = \overline{E}_\gamma$.

9.31.

Let class $Q = P$.

No two sets $E_\alpha = 1, \ldots, p$ of 9.29 have a common member.

By 9.30 there are γ, σ such that

$$\bar{E}_\gamma = E_\alpha \,, \quad \bar{E}_\sigma = E_\beta .$$

By the definition of 3.2,

$$\bar{E}_\gamma \cap E_\sigma = E_\alpha \cap E_\beta = 0.$$

9.32.

Let class $Q = p$.

The sets $E_\alpha, \alpha = 1, \ldots, p$ and F of 9.29 are unique, apart from the numbering.

For by 3.4, Theorem 3, the $\bar{E}_\beta$ are unique, apart from the numbering, and there is a β such that $\bar{E}_\beta = E_\alpha$, by 9.30; and if $E_\alpha \neq E'_\alpha$, $\bar{E}_{\beta'} \neq \bar{E}_\beta$, (where $\bar{E}_{\beta'} = E'_\alpha$)
The set $F = E - \bigcup_{\alpha=1}^{p} E_\alpha$, is then also unique.

9.33.

If P_α, $P_\alpha = [p_{ij}]$, $i, j \in \bar{E}_\alpha$, is a P-matrix, then $P_{\alpha\beta} = 0 \quad \beta \neq \alpha$, where the $\bar{E}_\alpha$ (and $P_{\alpha\beta} : [p_{ij}]$ $i, \in \bar{E}_\alpha$, $j \in \bar{E}_\beta$) are defined as in 3.2 (with $\bar{E}_\alpha$ for E_α, $P_{\alpha\beta}$ for $A_{\alpha\beta}$).

Let $i \in \bar{E}_\alpha$, $j \in \bar{E}_\beta$, $\beta \neq \alpha$.

By 28, as P_α is a P matrix, $p_{ij} = 0$.

Hence $P_{\alpha\beta} = 0$.

9.34.

If $P_\alpha : [p_{ij}]$, $i, j \in \bar{E}_\alpha$ is a P - matrix, and $\beta \neq \alpha$, then $R_{\alpha\beta} = R_{\beta\alpha} = 0$, where $R_{\beta\alpha}$ is defined by 3.3.

For $R_{\alpha\beta} = \sum r_{\alpha_1\alpha_2} r_{\alpha_2\alpha_3} \ldots\ldots r_{\alpha_{s-1},\alpha_s}$,

where $\alpha_1 = \alpha$, $\alpha_s = \beta$.

But $r_{\alpha_1\alpha_2} = 0$ for all $\alpha_2 \neq \alpha_1$, $1 \leq \alpha_2 \leq k$, as

$P_{\alpha_1\alpha_2} = 0$, by 9.33.

Hence $R_{\alpha\beta} = 0$, $\beta \neq \alpha$.

Similarly, by considering $r_{\gamma\alpha}$, we may prove

$R_{\beta\alpha} = 0$.

9.35.

An arrangement of the $\bar{E}_\alpha$, say E'_α, $\alpha = 1, \ldots, k$ satisfies the conditions of 3.2, Theorem 1, when

$$P'_{\alpha\beta} = 0 \quad \text{for} \quad \alpha < \beta,$$

where $P'_{\alpha\beta} = [p_{ij}]$, $i \in E'_\alpha$, $j \in E'_\beta$.

Let $R'_{\alpha\beta}$ be defined for the $P'_{\alpha\beta}$ as $R_{\alpha\beta}$ is for the $P_{\alpha\beta}$.

By 3.4, Theorem 2, E'_α, $\alpha = 1, \ldots, k$ satisfy the conditions of 3.2, Theorem 1, if and only if

$$R'_{\alpha\beta} = 0, \quad \text{for} \quad \alpha < \beta.$$

9.36.

There is an arrangement of the $\bar{E}_\alpha$, E'_α, $\alpha = 1, \ldots, k$ say, satisfying the conditions of 3.2, Theorem 1, such that $E'_\alpha = E_\alpha$, $\alpha = 1, \ldots, p$, where the E_α are the sets of

9.29.

Let us suppose, as we may by 9.30,

that $E_\alpha = \bar{E}_{\beta_\alpha} \quad \alpha = 1, \ldots, p.$

Let the remaining $\bar{E}_\alpha$ be denoted by $\bar{E}_{\gamma_\alpha}, \alpha = 1, \ldots, k - p,$
where $\gamma_\phi > \gamma_\alpha$, when $\phi > \alpha$.

Let $E'_\alpha = E_{\sigma_\alpha}, \alpha = 1, \ldots, k.$

and put $\sigma_\alpha = \beta_\alpha, \quad \alpha = 1, \ldots, p$

and $\sigma_\alpha = \gamma_{(\alpha - p)} \quad \alpha = p + 1, \ldots, k.$

If $\alpha \leq p$ and $\phi > \alpha$,

then $R'_{\alpha\phi} = R_{\beta_\alpha \sigma_\phi} = 0$ by 34, ~~as $\gamma_\phi > \beta_\alpha$~~.

~~as~~ as $E'_\alpha = E_\alpha$, and so $P'_\alpha \left[= P'_{\alpha\alpha}\right]$ is a P - matrix.

If $\alpha \geq p$ and $\phi > \alpha$,

then $R'_{\alpha\phi} = R_{\sigma_\alpha \sigma_\phi} = R_{\gamma(\alpha-p)\gamma(\phi-p)} = 0$,

as $\gamma(\alpha - p) < \gamma(\phi - p)$ and the $\bar{E}_\alpha$ satisfy the conditions of 3.2, Theorem 1.

Hence by 3.4, Theorem 2, the E'_α satisfy the conditions of 3.2, Theorem 1,

and $E'_\alpha = E_\alpha, \quad \alpha = 1, \ldots, p.$

9.37.

There is a conjugate permutation of rows and columns of Q so that after the permutation

$$Q = \left[\begin{array}{ccccc|c} P_1 & & & & & \\ & P_2 & & & & \\ & & P_3 & & & \\ & & & \ddots & & \\ & & & & P_p & \\ \hline \cancel{Q_2} & Q_2 & & & & Q_1 \end{array}\right]$$

where P_α, $\alpha = 1, \ldots, p$ are irreducible p - matrices, $p =$ class Q, and class $Q_1 = 0$.

By 9.36 there is an arrangement E'_α of the $\bar{E}_\alpha$ satisfying the conditions of 3.2, Theorem 1, such that $E'_\alpha = E_\alpha$, $\alpha = 1 \ldots p$.

Hence ~~by~~ put~~ting~~ the $i \in E_1$ first by a conjugate permutation, next the $i \in E_2$, and so on to $i \in E_p$. Finally put last the $i \in F = E - \mathcal{G}$.

Then if $P_\alpha = [p_{ij}]$, $i, j \in E_\alpha$, $Q_1 = [p_{ij}]$ $i, j \in F$, we have the diagonal matrices as indicated.

By 9.28, $p_{ij} = 0$, where $i \in E_\alpha$, $j \notin E_\alpha$, and by 9.29, class $Q_1 = 0$.

9.38.

If P is a $\mathfrak{P}$ - matrix, and F is non-empty, then $Q_2 \neq 0$.

Suppose that $Q_2 = 0$.

Let $i \in F$.

Then $\sum_{j \in E} p_{ij} = \sum_{j \in F} p_{ij} = 1$.

Hence Q_1 is a $\mathfrak{P}$ - matrix.

By 9.27, class $Q_1 \geq 1$.

But this contradicts 9.37.

Hence $Q_1 \neq 0$, if F is non-empty.

PART 5. Latent Roots and Vectors of Q - Matrices.

9.39.

Let class Q = 0 . Then if λ is a latent root of Q, $|\lambda| < 1$.

Suppose λ, $|\lambda| = 1$, is a latent root of Q, and y' its associated latent row vector.

Let $i \in F_1$, when $y_i \neq 0$,

$i \in F_2$, when $y_i = 0$.

Then $|y_j| = \sum_{i \in E} |y_i| \; p_{ij}$, by 3?,

$= \sum_{i \in F_1} |y_i| \; p_{ij}$.

Hence $\sum_{j \in F_1} |y_j| = \sum_{i,j \in F_1} |y_i| \; p_{ij} = \sum_{i \in F_1} |y_i| \; p_i$,

where $p_i = \sum_{j \in F_1} p_{ij}$.

As $y_i \neq 0$ when $i \in F_1$, it follows that $p_i = 1$, $i \in F_1$.

Hence $P = [p_{ij}]$, $i, j \in F_1$, is a P - matrix,

Whence by 9.27 there is an irreducible principal $P_2 \prec P_1$.
But then also $P_2 \prec Q$ and so class $Q \geq 1$. This is a contradiction and the result follows.

9.40.

Let class Q = p . Then the latent roots of unit modulus of Q are L_{r_1} , L_{r_2} ,..., L_{r_p}, where mod P_α $= r_\alpha$, and L_r is the set of r -th roots of unity. Each latent root λ has a multiplicity equal to the number of times it occurs in L_{r_α}, $\alpha = 1, \ldots, p$.

The latent roots of Q are those of P_1, $P_2, \ldots P_p, Q_1$. But Q_1 has no latent roots of unit modulus, and P_α has the set L_{r_α} when mod $P_\alpha = r_\alpha$, by 16. The result then follows.

9.41.

The multiplicity of $\lambda = 1$ is class Q.

Suppose class $Q = p$.

Then $1 \in L_{r_\alpha}$, $\alpha = 1, \ldots, p$ as $r_\alpha = \bmod P_\alpha \geq 1$.

By 40, the multiplicity of 1 is p.

9.42.

Let $y^{(\alpha)'}$ be the latent row vector of P_α associated with $\lambda \in L_{r_\alpha}$.

Then
$$y_i^{\alpha'} = y_i^{(\alpha)'}, \quad i \in E_\alpha,$$
$$y_i^{\alpha'} = 0, \quad i \notin E_\alpha,$$

are the elements of a latent row vector of Q associated with λ.

Let $j \in E_\alpha$;

then $\sum\limits_{i \in E} y_i^\alpha \, p_{ij} = \sum\limits_{i \in E_\alpha} y_i^{(\alpha)} \, p_{ij} = \lambda_i \, y_i = \lambda_i \, y_i$,

as by 28, $p_{ij} = 0$, $i \in E_\alpha$, $j \notin E_\alpha$.

Let $j \notin E_\alpha$.

Then
$$\sum_{i \in E} y_i^\alpha \, p_{ij} = \sum_{i \in E_\alpha} y_i^{(\alpha} \, p_{ij} = 0.$$

In 42 we did not assume that λ was a latent root of Q. This would follow as a corollary.

9.43.

Let $\lambda \in L_{r_\alpha}$ when $\alpha \in S_\lambda$. The y^α, $\alpha \in S_\lambda$, as defined in 42, are linearly independent.

If $\alpha \in S_\lambda$, then y^α is defined by 42.

Let $y = \sum_{\alpha \in S_\lambda} \xi_\alpha y^\alpha$.

Now $y_i^\beta = 0$, $i \in E_\alpha$, $\beta \neq \alpha$,

$y_i^\alpha = y_i^{i\alpha} \neq 0$, $i \in E_\alpha$, by 5.

Hence if $y_i = 0$ when $i \in E_\alpha$, then $\xi_\alpha = 0$.

It follows that if $y_i = 0$ then $\xi_\alpha = 0$, $\alpha \in S_\lambda$,

and this is equivalent to the above statement.

9.44.

Let S_λ be defined as in 43, and suppose S_λ has ρ members. Then the ρ y^α, as defined in 42, form a complete set of latent row vectors associated with λ

Cf. 6.5 .

In 43, we proved that the y^α, $\alpha \in S_\lambda$ were linearly independent. The multiplicity of λ is ρ, by 40. Hence, by 6.7, the y^{α}, $\alpha \in S_\lambda$ form a complete set of latent row vectors associated with λ.

9.45.

Let λ, $|\lambda| = 1$, be a latent root of Q, and let S_λ defined as in 43, have ρ members. Then there are ρ classical canonical submatrices of order 1 associated with λ.

(By 40 and 16, there is an α such that $\lambda \in L_{r_\alpha}$. Hence S_λ may be defined by 43.)

By 49, the multiplicity of λ is ρ.

By 44, there are ρ linearly independent (primary) latent row vectors associated with λ. Hence, by Ch. 3, theorem 1, all the classical canonical submatrices associated with λ are of order 1. Since $\rho = \sum_{j=1}^{p'} \nu_j'$, where p' is the number of classical canonical submatrices associated with ~~and~~ λ, and ν_j', are their orders, $p' = \rho$.

9.46.

Let $|\lambda| = 1$. Then $(\lambda I - Q_1)$ is non-singular.

For by 29, class $Q_1 = 0$, and hence, by 39, Q_1 has no latent root of unit modulus. Hence $(\lambda I - Q_1)$ is non-singular.

9.47.

Let $x^{(\alpha)}$ denote the latent column vector of P_α associated with $\lambda \in L_{r_\alpha}$.

Then $x_i^\alpha = x_i^{(\alpha)}$, $\quad i \in E$,

$x_i^\alpha = 0$, $\quad i \in G$, $i \notin E_\alpha$,

and $x_i^\alpha = x_i^{[\alpha_2]}$, $\quad i \in F$,

is a latent vector of Q associated with λ

where $x_i^{[\alpha_1]} = x_i^\alpha$, defined for $i \in G$

and $\underline{x^{[\alpha_2]} = (\lambda I - Q_1)^{-1}} Q_2 \, x^{[\alpha_1]}$.

Let $i \in E_\alpha$.

Then $\lambda x_i^{(\alpha)} = \sum_{j \in E_\alpha} p_{ij} x_j^\alpha$.

But $p_{ij} = 0$, $i \in E_\alpha$, $j \notin E_\alpha$, by 28.

Hence $\lambda x_i^\alpha = \sum_{j \in E} p_{ij} x_j$, $\quad i \in E_\alpha$.

Let $i \in G$, $i \notin E_\alpha$.

Then $0 = \lambda x_i = \sum_{j \in E} p_{ij} x_j$,

for if $i \in E_\beta$, $\beta \neq \alpha$, $\sum p_{ij} x_j^\alpha = 0$, $j \in E$.

Let $i \in F$.

Then $\lambda x_i^\alpha = \sum_{j \in E} p_{ij} x_j^\alpha = \sum_{j \in G} p_{ij} x_j^\alpha + \sum_{j \in F} p_{ij} x_j^\alpha$.

Hence

$$\lambda x^{[\alpha_2]} = Q_2 x^{[\alpha_1]} + Q_1 x^{[\alpha_2]},$$

whence $x^{[\alpha_2]} = (\lambda I - Q_1)^{-1} Q_2 x^{[\alpha_1]}$.

9.48.

Let x^α be defined as in 47. Then the x^α, $\alpha \in S_\lambda$, are linearly independent.

Let $x = \sum_{\alpha \in S_\lambda} c_\alpha x^\alpha = 0$.

Let $i \in E_\beta$, then $x_i = \sum c_\alpha x_i^\alpha = c_\beta x_i^\beta$, by 47.

By 9, $x_i^\beta = |x_i^{(\beta)}| = 1$, $i \in E_\beta$,

and hence $c_\beta = 0$.

We may thus prove $c_\alpha = 0$, $\alpha \in S_\lambda$, and the result follows.

9.49.

The latent column vectors x^α, $\alpha \in S_\lambda$, form a complete set of latent column vectors of Q associated with λ.

Let S_λ have ρ members. There are ρ linearly independent latent column vectors, by 48. But the multiplicity of λ is ρ by 40. Hence by 6.7 the x^α form a complete set of latent column vectors.

9.50.

When $\lambda = 1$, $x_i^{\alpha} \geq 0$, $i \in E$ for $\alpha = 1, \ldots, p$.
(As $1 \in L_{r_\alpha}$ $\alpha = 1, \ldots, p$, x^{α} is defined for $\alpha = 1, \ldots, p$.)

By 47 and 9,

$$x_i^{\alpha} = 1 \quad , \quad i \in E_{\alpha} .$$

$$x_i^{\alpha} = 0 \quad , \quad i \in G \ , i \in E_{\alpha} .$$

By 2, 29, 40 the greatest non-negative latent root of Q_1 is smaller than 1, Hence by Theorem 1, 7.3, $(I - \cancel{\lambda} Q_1)^{-1} \supset 0$.
and hence $(I - Q_1)^{-1} Q_2 \ x^{[\alpha]} \supseteq 0$.
Hence $x_i \geq 0$, $i \in F$.
and the result follows.

9.51.

If $y^{\alpha'}$ is the latent row vector of $\lambda \in L_{r_\alpha}$ defined in 42, then $|y_i^{\alpha}|$ is independent of the particular $\lambda \in L_{r_\alpha}$.

This follows immediately by the definition of $y^{\alpha'}$ and 19.

9.52.

If x^{α} is the latent column vector of $\lambda \in L_{r_\alpha}$ defined in 47, then $|x_i^{\alpha}|$, $i \in G$, is independent of $\lambda \in L_{r_\alpha}$.
This follows immediately by the definition of x^{α} and 9.

When $i \in F$, x_i^{α} is not in general independent of λ, e.g.

$$P = \begin{bmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Then $E_1 = (1, 2) = G$, $F = (3)$, $L_{r_1} = (1, -1)$.

The vector $x = \{1, 1, 1\}$ is the latent column vector associated with 1, but the latent column vector associated with -1 is $\{1, -1, \tfrac{1}{3}\}$.

9.53.

Let S_λ have s members. Then the sets x^α, $\alpha \in S_\lambda$ and $y^\alpha, \alpha \in S_\lambda$ are biorthogonal, viz.

$$y^{\beta'} x^\alpha > 0 \quad , \text{ if } \quad \beta = \alpha.$$
$$y^{\beta'} x^\alpha = 0 \quad , \text{ if } \quad \beta \neq \alpha.$$

We have $y^{\alpha'} x^\alpha = \sum_{i \in E} y_i^\alpha x_i^\alpha = \sum_{i \in E_\alpha} y_i^{\alpha'} x_i^\alpha$, by 42 and 47.
And the result follows for $\beta = \alpha$, as $y_i^\alpha > 0$, $x_i^\alpha > 0$, $i \in E_\alpha$.
If $\beta \neq \alpha$, $y_i^\beta = 0$, unless $i \in E_\beta$,

$x_i^\alpha = 0$, when $i \in E_\beta$.

Hence $y^{\beta'} x^\alpha = \sum_{i \in E} y_i^\beta x_i^\alpha = 0$.

9.54.

Summary of Part 5.

The latent roots λ, $|\lambda| = 1$ of a Q - matrix, are those of the irreducible P contained in Q. The multiplicity of λ equals the number of P_α of which *it* is a latent root. The classical canonical submatrices associated with λ are of order 1.

The vector y' is a latent row vector of Q associated with λ if and only if

$$y = \sum_{\alpha \in S_\lambda} c_\alpha y^\alpha$$

for some constants c_α, where $\alpha \in S_\lambda$ if and only if λ is a latent root of P_α and y^α is defined by 42.

The vector x is a latent column vector of Q associated with λ if and only if

$$x = \sum_{\alpha \in S_\lambda} c_\alpha x^\alpha$$

for some constants c_α and x^α is defined by 47.
The sets of vectors $y^{\alpha'}$, x^α, $\alpha \in S_\lambda$, are biorthogonal.

PART 6. Alternative Proofs of Results for Latent Row Vectors.

9.55.

In 42 we remarked that it was not necessary to assume that λ was a latent root of Q, to prove that y^α was a latent row vector associated with λ. Similarly it is not necessary to assume that the multiplicity of λ is ρ (when S_λ has ρ members) to prove that every latent row vector associated with λ is a linear combination of the y^α.
We shall show this in the following sections. In Part 6 we shall only assume sections 39, 42, 43 of Part 5, and the definitions of y^α, S_λ.

9.56.

When F is defined as in 29, and y' is a latent row vector of Q associated with λ, $|\lambda| = 1$ then $y_i = 0$, $i \in F$.

We have $p_{ij} = 0$, $i \in G$, $j \in F$ by 28,

for if $i \in G$, then $i \in E_\alpha$ for some α,

and $j \notin E_\alpha$ when $j \in F$.

Let $j \in F$.

Then $\sum_{i \in E} y_i \, p_{ij} = \sum_{i \in F} y_i \, p_{ij} = \lambda y_i$.

Hence $z' Q_1 = \lambda z'$,

when z_i is defined for $i \in F$, and $z_i = y_i$, $i \in F$.

But Q_1 has no latent root λ, $|\lambda| = 1$, by 39 and 29.

Hence $y_i = z_i = 0$, $i \in F$.

9.57.

If y' is a latent row vector associated with $\lambda, |\lambda|=1$ and $y_k \neq 0$ $k \in E_\alpha$, then $\lambda \in L_{r_\alpha}$ and $y_i = c_\alpha y_i^{(\alpha)}$, $i \in E_\alpha$, where $y^{(\alpha)'}$ is the latent row vector of P_α associated with λ.

Let $j \in E_\alpha$, $\lambda y_j = \sum_{i \in E} y_i p_{ij} = \sum_{i \in E_\alpha} y_i p_{ij}$, by 28.

If $y_k \neq 0$, $k \in E_\alpha$, the y_i, $i \in E_\alpha$ are the elements of a latent row vector of P_α associated with λ.

Hence, $\lambda \in L_{r_\alpha}$, and as the latent row vector of P_α associated with λ is unique,

$$y_i = c_\alpha y_i^{(\alpha)}, \quad i \in E_\alpha.$$

9.58.

If y' is a latent row vector associated with λ and $\alpha \in S_\lambda$ when $\lambda \in L_{r_\alpha}$,

then $$y = \sum_{\alpha \in S_\lambda} c_\alpha y^{\alpha}$$

for some constants c_α.

By 57, $y_i = 0$, if $i \in E_\beta$ and $\lambda \notin L_{r_\beta}$, i.e. $\beta \notin S_\lambda$.

and $\sum_{\alpha \in S_\lambda} c_\alpha y_i^{\alpha} = 0$, by 42, $\beta \notin S_\lambda$, By 57, $y_i = c_\alpha \cdot y_i^{c\alpha}$ if $i \in E_\alpha$, and $\alpha \in S_\lambda$,

and $c_\alpha y_i^{c\alpha} = c_\alpha y_i^\alpha = \sum_{\alpha \in S_\lambda} c_\alpha y_i^\alpha$, by 42.

By 56, $y_i = 0$, $i \in F$,

and $\sum_{\alpha \in S_\lambda} c_\alpha \cdot y_i^\alpha = 0$, by 42.

The result follows.

9.59.

If $y = \sum_{\alpha \in S_\lambda} c_\alpha y^\alpha$, then y' is a latent row vector of Q associated with λ, $|\lambda| = 1$.
As any linear combination of latent row vectors is a row vector, the result follows immediately.

9.60.

The vector y' is a latent row vector associated with λ, if and only if $y = \sum_{\alpha \in S_\lambda} c_\alpha y^\alpha$.
By 58 and 59.

9.61.

We have proved the result enunciated in 55 without assuming anything about the multiplicity of λ.
It should be noted that the result of 40 does not follow immediately by this method, as we should first have to prove that the classical submatrices associated with λ are 1×1. This may also be done.

9.62.

Let y' be a latent row vector of Q associated with λ. There is no vector z such that $z' Q = \lambda z' + y'$.
By 60,

$$y = \sum_{\alpha \in S_\lambda} c_\alpha y^\alpha .$$

Suppose $z' Q = \lambda z' + y'$.
Let $j \in F$. Then by 42, $y_j = 0$.
Hence $\sum_{i \in E} z_i p_{ij} = \sum_{i \in F} z_i p_{ij} = \lambda z_j$.
It follows that the vector $\bar{z}$, where $\bar{z}_i$ is defined for

$i \in F$ and $\bar{z}_i = z_i$, $i \in F$, satisfies

$$\bar{z}' Q_1 = \lambda \bar{z} .$$

But by 29 and 39, λ is not a latent root of Q_1.

Hence $z_i = \bar{z}_i = 0$, $i \in F$.

As $y \neq 0$, there is some $\alpha \in S_\lambda$ such that

$$c_\alpha \neq 0 .$$

Let $c_\beta \neq 0$.

Then $\sum_{i \in E} z_i \; p_{ij} = \sum_{i \in E_\beta} z_i \; p_{ij} = \lambda z_j + y_j = \lambda z_j + c_\beta y_j^\beta$

when $j \in E_\beta$.

Hence if z_i^β is defined for $i \in E_\beta$, and $z_i^\beta = z_i / c_\beta$, ~~if~~ when $i \in E_\beta$, ~~then~~

~~then~~ $$z^{(\beta)'} P_\beta = \lambda z^{(\beta)'} + y^{(\beta)'} .$$

By 22, λ is a single latent root of P_β and the classical canonical submatrix of P_β associated with it is of order 1. Hence by 6.5, theorem 2, there is no vector $z^{(\beta)}$ satisfying

$$z^{(\beta)'} P_\beta = \lambda z^{(\beta)'} + y^{(\beta)'} .$$

We deduce the required result.

9.63.

The classical canonical submatrices of Q associated with λ are of order 1. If some submatrix is of order 2, then by 6.4 there is a latent row vector z' such that

$$z' Q = \lambda z' + y' .$$

But by 62 this is impossible and the result follows.

9.64.

The latent column vectors may be treated in a similar way.

PART 7. Vectors Associated with Sets of Latent Roots of Unit Modulus.

9.65.

By 19, we may associate a row vector $v' > 0$ with set L_r of latent roots of unit modulus of an irreducible P , such that, if y' is a latent row vector of P associated with $\lambda \in L_r$, $|y_i| = v_i$, $i \in E$ (or in the notation of chapter 4 $y^* = v$).

Correspondingly we shall say that a set L of latent roots of unit modulus of Q forms an L - set when we may associate with L a row vector $v' > 0$ such that, if y is a latent row vector of Q associated with $\lambda \in L$,
$|y_i| = v_i$, $i \in E$.

An L - set, L , will be called complete with respect to an associated vector v' when the facts that λ, $|\lambda| = 1$ is a latent root of Q , y' associated with λ and $|y_i| = v_i$, $i \in E$ imply $\lambda \in L$.

An L - set complete with respect to some v', may be called a "complete L - set".
Of course there may be two vectors associated with L , such that L is complete with respect to one, but not the other.

9.66.

Let $v_i^\alpha = |y_i^\alpha|$, where y^α is defined as in 42.
Then if L is a set of latent roots of unit modulus and $L \prec L_{r_\alpha}$ when $\alpha \in H$, where H is a non-empty subset of $(1,2,\dots,p)$, L is an L - set of Q and v', $v = \sum_{\alpha \in H} k_\alpha v^\alpha \supset 0$, $k_\alpha \geqslant 0$, is an associated vector.

If $\lambda \in L$ and $y_\lambda (y \neq 0)$, has elements

$$y_i = c_\alpha y_i^\alpha, \quad i \in E_\alpha, \text{ when } L \prec L_{r_\alpha} \text{ (or } \alpha \in H),$$

$$y_i = 0, \quad i \in E_\alpha, \quad L \not\prec L_r \text{ (or } \alpha \notin H),$$

$$y_i = 0, \quad i \in F,$$

then y_λ' is a latent vector associated with λ, by 59, as $L \prec L_{r_\alpha}$ implies $\lambda \in L_{r_\alpha}$.

Putting $|c_\alpha| = k_\alpha$, $|y_i| = v_i$ we have $v = \sum k_\alpha v^\alpha$,

$$v_i = |y_i| = k_\alpha v_i^\alpha, \quad i \in E_\alpha, \quad \alpha \in H,$$

$$v_i = 0, \quad i \in E_\alpha, \quad \alpha \notin H,$$

$$v_i = 0, \quad i \in F.$$

But $v_i^\alpha = |y_i^\alpha| = y_i^{(\alpha}$ as in 51,

and thus v_i^α is independent of the particular $\lambda \in L$.
Hence, as $v_i = k_\alpha v_i^\alpha$ or $v_i = 0$, it follows that v_i, $i \in E$, is also independent of the particular $\lambda \in L$.

Hence L is an L - set, and v' is associated with L.
The result follows, as $v = \sum_{\alpha \in H} k_\alpha v^\alpha \geqslant 0$, $k_\alpha \geq 0$ and $v \neq 0$.

9.67.

If v' is a vector associated with an L - set L,

$$v_i = 0 \;, \quad i \in E_\alpha, \text{ when } \quad L \not< L_{r_\alpha}$$

and $v_i = 0 \;, \; i \in F$.

For if $L \not< L_r$ there is an $\lambda \in L_{r_\alpha}$, such that $\lambda \notin L$.

Hence by 58 and 42,

$$y_i = 0 \;, \quad i \in E_\alpha \;, \quad L \not< L_{r_\alpha} \;,$$

and, $y_i = 0 \;, \; i \in F$,

for <u>every</u> latent row vector y' associated with λ.

If v' is the vector associated with L, there is some latent row vector associated with $\lambda \in$ L, such that

$$|y_i| = v_i \;, \quad i \in E.$$

Hence $v_i = 0 \;, \quad i \in E_\alpha, \quad L \not< L_{r_\alpha} \;,$

$$v_i = 0 \;, \quad i \in F.$$

9.68.

The set L of latent roots of unit modulus of Q , is an L - set if and only if there is an α such that $L < L_{r_\alpha}$.

By 66, if $L < L_{r_\alpha}$, L is an L - set.

If **L** is an L - set there is a (weakly) positive $v \geq$ associated with L.

By 67, $v_i = 0$, when $i \in E_\alpha$, $L \not< L_r$, or $i \in F$.

Hence if $\lambda \notin L_{r_\alpha}$, $\alpha = 1, \ldots, p$, then $v = 0$.

Thus if L is an L - set, there is an α such that $L \prec L_{r_\alpha}$.

9.69.

The set consisting of the latent root $\lambda = 1$ is an L - set.

For any α, $1 \leq \alpha \leq p$, $1 \in L_{r_\alpha}$. The result follows by 68.

6.70.

The vector v' is associated with an L - set L, if and only if

$$v = \sum_{\alpha \in H} k_\alpha v^\alpha \supset 0, \qquad k_\alpha \geq 0, \quad v \neq 0$$

where $\alpha \in H$ when $L \prec L_{r_\alpha}$.

By 66, if $v = \sum_{\alpha \in H} k_\alpha v^\alpha \supset 0$, then v' is associated with L_{r_α}.

Let v' be associated with L.

Then $v_i = 0$, $i \in E_\alpha$, if $L \not\prec L_{r_\alpha}$, (i.e. if $\alpha \notin H$), and $v_i = 0$, $i \in F$, (by 67).

Suppose $\alpha \in H$, and $\lambda \in L$. Then there is a latent row vector associated with λ, such that

$$|y_i| = v_i, \qquad i \in E_\alpha.$$

But by 58 and 42, $y_i = c_\alpha y_i^\alpha$, $i \in E_\alpha$.

Hence $v_i = k_\alpha v_i^\alpha$, $i \in E_\alpha$ with $k_\alpha \geq 0$.

It follows that $v = \sum_{\alpha \in H} k_\alpha v^\alpha \supset 0$, $k_\alpha \geq 0$.

9.71.

The set, L, of latent roots of unit modulus is a complete L - set if and only if there is a subset K of $(1, 2, \ldots, p)$ such that $L = L_t$ where $t = \underset{\alpha \in K}{\text{g.c.d.}}\, r_\alpha$ (the greatest common divisor of r_α, for $\alpha \in K$).

The set L_t is complete with respect to $v = \sum_{\alpha \in K} k_\alpha v^\alpha$, $k_\alpha > 0$.

Let L be an L - set, and let v' be an associated vector.

By 70,
$$v = \sum_{\alpha \in K} k_\alpha v^\alpha, \quad k_\alpha > 0,$$

where K is a subset of the set H, defined as in 66.

Hence by definition of H, $L \prec L_{r_\alpha}$ when $\alpha \in K$.

and so $L \prec \bigcap_{\alpha \in K} L_{r_\alpha}$

(the intersection of L_{r_α} for $\alpha \in K$).

But as L_{r_α} is the set of r_α-th roots of unity,

$\bigcap_{\alpha \in K} L_{r_\alpha} = L_t$ as defined above.

Thus $L \prec L_t$.

By 66, the set L_t is an L - set, and $v = \sum_{\alpha \in K} k_\alpha v^\alpha$ is a vector associated with it.

Let $\lambda \in L_t$. Then by 68, λ is an L - set, and $v = \sum_{\alpha \in K} k_\alpha v^\alpha$, $k_\alpha > 0$ is a vector associated with the L - set λ.

Hence the L - set L is complete (and hence any set L is complete) if and only if $L = L_t$ and L_t is complete with respect to v'.

9.72.

The set of latent roots L_{r_α} is complete with respect to its associated vector v^α .

If the set K of 70 consists of α , then $L_t = L_{r_\alpha}$ and $v = v^\alpha$.

The result then follows.

CHAPTER 10.

10.1.

In this chapter we shall consider the classical canonical submatrices associated with the non-negative latent root of maximum modulus of a (reducible) weakly positive matrix P and also the set of generalized latent vectors associated with that root.

Suppose P in the normal form of Chapter 3.

$$P = \begin{bmatrix} P_{11} & & & & \\ P_{21} & P_{22} & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ P_{k1} & P_{k2} & \cdot & \cdot & \cdot & P_{kk} \end{bmatrix}$$

and let ρ be the greatest latent root of P.

Let P_{ii} be a square matrix of order n_i. It will generally be more convenient to consider $A = \rho I - P$. Then A is singular.

A is also in normal form,

$$A = \begin{bmatrix} A_{11} & & & & \\ A_{21} & A_{22} & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ A_{k1} & A_{k2} & \cdot & & & A_{kk} \end{bmatrix}.$$

If λ is any latent root of P, $|\lambda| \leq \rho$.

Hence the real part of the latent root $\rho - \lambda$ of A satisfies

$\Re(\rho - \lambda) \geq 0.$

Thus A is a singular matrix, all of whose non-zero latent roots have positive real parts. Let us denote the singular A_{ii} by $A_{\alpha_1\alpha_1}$, $A_{\alpha_2\alpha_2}$, ..., $A_{\alpha_c\alpha_c}$.

By Collatz's theorem (cf. 8.2) applied to P_{ii},

$$\rho \geq \rho_i \geq \min_{(j \in n_i)} (P_{ii}\, e^i)_j \geq p_{kk}$$

where ρ_i is the greatest latent root of P_{ii}, e^i is the n_i vector $e^i = \{1, 1, \ldots\ldots 1\}$, and p_{kk} is the j-th element of P_{ii}. (of n_i elements)

Let E_i be the set of integers k such that p_{kk} is an element of P_{ii}, and let $j \in E_i$.

By 8.3

$$\rho = \rho_i = \min_{j \in n_i} (P_{ii}\, e^i)_j = p_{kk},$$

if and only if

$$\rho\, e^i = P_{ii}\, e^i = p_{kk}\, e^i.$$

Hence $\quad P_{jk} = 0, \quad k \neq j, j \in E_i, \quad k \in E_i. \qquad (1)$

But P_{ii} is irreducible and hence (1) can not hold. This implies that E_i consists of k, i.e. P_{kk} is the 1 x 1 matrix $P_{ii} = [p_{kk}]$.

It follows that the diagonal elements a_{jj} of A are positive, except that $a_{jj} = 0$ is possible when $[a_{jj}]$ is one of irreducible submatrices of A. The non-diagonal elements of A satisfy $a_{ij} = -p_{ij}$, and hence $a_{ij} \leq 0, \quad i \neq j$.

10.2.

As $\rho \geq \rho_i$ and P_{ii} is irreducible,

$$\operatorname{adj} (\rho I - P_{ii}) = \operatorname{adj} A_{ii} > 0,$$

by 7.4, theorem 3.

Also $|A_{\alpha_i \alpha_i}| = 0, \quad i = 1, \ldots\ldots, l.$

The following theorem is of fundamental importance in the development of our results. Clearly the irreducible $A_{\alpha_i \alpha_i}$ satisfy the conditions imposed on the matrix of the theorem.

Theorem 1.

Let $|A| = 0$, adj $A > 0$, $z \geqq 0$.

There is a vector w for which $A w = z$, if and only if $z = 0$.

Proof. Suppose $A w = z$.

Then adj $A \cdot z$ = adj $A \cdot A w = 0$.

But by 4.2.6, if $z > 0$, adj $A \, z > 0$,

whence $z = 0$.

There is a latent vector x such that $A x = 0$, as $|A| = 0$.

This proves the second part of the theorem.

We may incidentally note that adj $A > 0$ only if A is irreducible. For if A is reducible, and adj A is partitioned similarly to A, $(\operatorname{adj} A)_{ij} = 0 \quad j > i$, as adj A is a function of A, (c.f. 5.7.)

The equation

$$A w = z, \quad w \neq 0 \qquad (2)$$

may be written

$$z = \sum_{i=1}^{n} w_i \, a_{*i}.$$

Hence z is linearly dependent on the columns of A if and only if there is a w satisfying (2). Suppose $A = \rho I - P$, where P is an irreducible positive matrix, ρ its largest non-negative latent root.

Then $|A| = 0$, and adj $A > 0$.

There is a latent vector x of A such that $A x = 0$, and $x > 0$.

By theorem 1, x is linearly independant of the columns of A. ((By 6,6 this is another proof that the classical canonical submatrix associated with x is of order 1×1). But as x is the only latent vector associated with the latent root 0 of A, the rank of A is $n - 1$. Hence A contains $n - 1$ linearly independent columns. We may go further than this. As adj $A > 0$, any principal minor of order $n - 1$ of A is non-zero. Hence its columns are linearly independent, and, a fortiori, the columns of A containing it are linearly independent. We have proved that any $n - 1$ columns of A are linearly independent.

Let y be linearly independent of the columns of A. Then a_{*1}, a_{*2}, . . . , a_{*n-1}, y, are a set of n linearly independent vectors, and as any vector of n elements can be expressed in terms of a set of n linearly independent vectors, we have for any z,

$$z = \sum_{i=1}^{n-1} c_i a_{*i} + c_n y .$$

This property of the set a_{*1}, $a_{*2}, \dots, a_{*n-1}, y$, is often expressed by: "the set forms a basis for the vectors z".

By theorem 1 if $y > 0$, then y is linearly independent of the columns of A. Putting $c_i = w_i$, $i = 1, \ldots, n-1$, $c_n = -c$, we immediately obtain Corollary 1.

Corollary 1. Let A be singular, $\operatorname{adj} A > 0$, $y \supset 0$, and z be any vector. Then there is a vector, w, s.t. $w_n = 0$, and a scalar c, such that

$$Aw = -cy + z.$$

(Of course, if A is non-singular, the corollary is trivially true for $c = 0$, or indeed any c.)

10.3.

Now let us suppose that A is singular, $\operatorname{adj} A > 0$ and that $z \supset 0$.

If $\qquad Aw = -cy + z$

(we do not need to insist on $w_n = 0$)

and $\qquad c \leq 0,$

then $\qquad -cy + z \supset 0.$

But this is not possible by Theorem 1. Hence $c > 0$. We have proved Corollary 2.

Corollary 2. Let A be singular, $\operatorname{adj} A > 0$, y and z be (weakly) positive vectors. There is a vector w and a scalar c such that

$$Aw = -cy + z.$$

The scalar c is positive.

10.4.

We now investigate solutions of $Ax = 0$. Let us partition

$x = \{x_1, x_2, \ldots, x_k\}$, conformably with A.

Then $Ax = \{A_{11} x_1, \sum_{j=1}^{2} A_{2j} x_j \quad \sum_{j=1}^{k} A_{kj} x_j\}$

It is useful to introduce the concept of partial solutions of $Ax = 0$. (We shall give definitions which do not assume $A \supset 0$). Suppose that

$$\left.\begin{array}{l} x_1 = x_2 \quad = x_{h-1} = 0 \\ x_h \neq 0 \end{array}\right\}. \qquad (3)$$

Then $(Ax)_1 = (Ax)_2 = \cdots = (Ax)_{n-1} = 0$,

where $(Ax)_i$ is the i-th vector component of Ax.

If x satisfies (3) and Ax also satisfies

$$(Ax)_1 = (Ax)_2 \quad = (Ax)_j = 0 \qquad (4)$$

for $j \geqslant h$, ...

we shall call x a partial $h - j$ solution of $Ax = 0$. A partial $h - k$ solution of $Ax = 0$, is a solution of $Ax = 0$, and will be called an h - solution. Evidently any solution is an h - solution for some h. We shall at this point introduce some other terminology. A vector satisfying (3) will be called an h - vector. Evidently every vector is an h - vector for some h. A vector for which only x_i $i = 1, \ldots, j$ is defined (and which may be regarded as conformal with $\bar{A} = [A_{ip}]$, $i, p = 1, \ldots, j$, and which in addition satisfies (3) will be called an $h - j$ vector.

10.5.

In this section it will not be assumed that $A \supset 0$. The singular A_{ii} will still be denoted by $A_{\alpha_i \alpha_i}$, $i = 1, \ldots, \ell$.

Lemma. 1. <u>If x is a partial solution of $Ax = 0$, there is an i such that x is an α_i-vector.</u>

Let x be an h - vector,

Then $(Ax)_h = A_{hh} x_h = 0$, and $x_h \neq 0$.

Hence A_{hh} is singular, and so there is such an i that $h = \alpha_i$.

This proves the lemma.

10.6.

We shall now denote any partial α_i - j solution of $Ax = 0$ by x^i and again assume that $A \geqslant 0$. It is clear that $x^i_{\alpha_i}$ is a latent vector of $A_{\alpha_i \alpha_i}$, associated with 0. Hence it is also the latent vector of $R_{\alpha_i \alpha_i}$ associated with its non-negative latent root of maximum modulus, ρ. But this latent vector is unique (cf. 8.ρ), and on proper normalization we have $x^i_{\alpha_i} \gg 0$.

We shall next prove a theorem which we shall often refer to.

Theorem 2. <u>There is a unique partial $\alpha_i - \alpha_{i+1} - 1 + 3$ solution x^i of $Ax = 0$ (apart from a multiplicative constant). The vector x^i satisfies</u>

$$(X^i)_{\alpha_{i+1}-1} \qquad x^i_j > 0, \quad \text{if} \quad R_{j\alpha_i} > 0,$$

$$x^i_j = 0, \quad \text{if} \quad R_{j\alpha_i} = 0,$$

<u>where $R_{j\alpha_i}$ is defined by 3.3.</u> $j = \alpha_i, \alpha_i + 1, \ldots, \alpha_{i+1} - 1.$

Let $x^i_h = 0, \quad h = 1, \ldots, \alpha_i - 1$

$$A_{\alpha_i\alpha_i} x^i_{\alpha_i} (Ax^i)_{\alpha_i} = 0 \qquad (5)$$

and $$A_{hh}\, x^i_h = z^i_h \qquad \alpha_i + 1 \leq h \leq \alpha_{i+1}, \qquad (6)$$

where $$z^i_h = -\sum_{j=1}^{h-1} A_{hj}\, x^i_j.$$

There is a unique (apart from a multiplicative constant) $x^i_{\alpha_i}$ satisfying (5), and $x^i_{\alpha_i} > 0$. As A_{hh} is non-singular, and $\alpha_i < h < \alpha_{i+1}$ we may successively solve (6) for $h = \alpha_i + 1$, $\alpha_i + 2, \ldots, \alpha_{i+1} - 1$. Hence there is a partial $\alpha_i - \alpha_{i+1} - 1$ solution of $Ax = 0$. If $x^i_j, j = 1, \ldots, h - 1 \geq \alpha_i$, are unique, (6) yields a unique x^i_h. As $x^i_j = 0$, $j = 1, \ldots, \alpha_i - 1$, and $x^i_{\alpha_i}$ is the unique vector satisfying (5), we see by induction that the $\alpha_i - \alpha_{i+1} - 1$ solution is unique.

Let us now prove our statement about the x^i_h. Suppose inductively that for $\alpha_i \leq h \leq \alpha_{i+1} - 2$:

$$(X_i)_h : \quad x^i_j > 0, \quad \text{if } R_{j\alpha_i} > 0,$$
$$x^i_j = 0, \quad \text{if } R_{j\alpha_i} = 0,$$
$$\text{for } j = \alpha_i, \alpha_i + 1, \ldots, h.$$

$$(Z_i)_h : \quad z^i_j > 0, \quad \text{if } R_{j\alpha_i} > 0,$$
$$z^i_j = 0, \quad \text{if } R_{j\alpha_i} = 0,$$
$$\text{for } j = \alpha_i + 1, \ldots, h + 1.$$

(We remember that we have either $R_{ji} > 0$, or $R_{ji} = 0$).

From (6) $$x^i_{h+1} = A^{-1}_{h+1\,h+1}\, z^i_{h+1},$$

and by 7.4, $A^{-1}_{h+1\,h+1} > 0$.

~~and~~ this yields,

$x^i_{h+1} > 0$, if $z^i_{h+1} > 0$ i.e. if $R_{h+1\alpha_i} > 0$,

$x^i_{h+1} = 0$, if $z^i_{h+1} = 0$, i.e. if $R_{h+1\alpha_i} = 0$,

by $(Z_i)_h$.

This is $(X_i)_{h+1}$.

By definition, $z^i_{h+2} = -\sum_{j=\alpha_i}^{h+1} A_{h+2j}\, x^i_j$.

As $A_{h+2j} \leq 0$,

$$z^i_{h+2} \geq 0, \tag{7}$$

and $z^i_{h+2} > 0$, if and only if there is a j, $\alpha_i \leq j \leq h+1$, for which $-A_{h+2j}\, x^i_j > 0$. But $x^i_j \geq 0$, $z^i_{h+2} > 0$, if and only if $-A_{h+2j} > 0$, and $x^i_j > 0$,

i.e. if and only if

$$r_{h+2j} > 0 \quad \text{and} \quad R_{j\alpha_i} > 0,$$

by definition of r_{h+2j} and $(X_i)_{h+1}$.

Hence $z^i_{h+2} > 0$, if and only if there is a j for which $r_{h+2j} R_{j\alpha_i} > 0$. There is such a $\underline{j}$ if and only if $R_{h+2\alpha_i} = \sum_{j=\alpha_i}^{h+1} r_{h+2j} R_{j\alpha_i} > 0$. Combining this with (7) we immediately obtain $(Z_i)_{h+1}$.

The proposition $(X_i)_{\alpha_i}$ is equivalent to $x^i_{\alpha_i} > 0$ as $R_{\alpha_i\alpha_i} = 1$, by definition, ~~and~~ this we have shown to be true. As $z^i_{\alpha_i+1} > 0$ or $z^i_{\alpha_i+1} = 0$ according as $A_{\alpha_i+1\alpha_i} < 0$ or $A_{\alpha_i+1\alpha_i} = 0$, i.e. according as $R_{\alpha_i+1\alpha_i} > 0$ or $R_{\alpha_i+1\alpha_i} = 0$, $(Z_i)_{\alpha_i}$ is clearly true.

By induction we obtain $(X_i)_{\alpha_{i+1}-1}$, and this is our theorem. We point out, for future use, that by induction we also obtain $(Z_i)_{\alpha_{i+1}-1}$.

(It may be noted that if we define $z_h^i = -\sum_{j=1}^{h-1} A_{hj} x_j^i$, $(Z_i)_{\alpha_i-1}$ is not true, but $(X_i)_h$ holds for $h = 1, 2, \ldots \alpha_i - 1$)

10.7.

<u>Theorem 3.</u> (i) There is a (weakly) positive α_i -solution of $Ax = 0$ if and only if

$$R_{\alpha_p \alpha_i} = 0 \quad \text{for} \quad p = i+1, i+2, \ldots\ldots \ell .$$

(ii) If $R_{\alpha_p \alpha_i} = 0$, for $p = i+1 \ldots\ldots \ell$ there is an α_i- vector x^i satisfying $Ax = 0$, such that

$$(X_i)_k \quad : \quad x_j^i > 0 \quad \text{if} \quad R_{j\alpha_i} > 0 ,$$
$$x_j^i = 0 \quad \text{if} \quad R_{j\alpha_i} = 0 ,$$
$$\text{for} \quad j = \alpha_i , \alpha_i + 1 , \ldots\ldots, k .$$

(We might have put "for $j = 1, 2, \ldots k$" but as we shall want to use $(Z_i)_h$, "$j = \alpha_i , \ldots\ldots k$ " seems preferable, cf. the end of 10.6.)

(a) Let us inductively suppose Q_h , where we put

Q_h ($\alpha_i \leq h < k$) ": There is a positive partial α_i - h solution of $Ax = 0$, if and only if $R_{\alpha_p \alpha_i} = 0$, for $\alpha_i < \alpha_p \leq h$, ~~and~~ If $R_{\alpha_p \alpha_i} = 0$ $\alpha_i < \alpha_p \leq h$, there is a positive partial α_i - h solution $\{x_j^i\}$ satisfying $(X_i)_h$."

As in 10.6 we may prove that $\{x_j^i\}$ satisfies $(Z_i)_h$, $h \geq \alpha_i$

when $\quad z^i_h = -\sum_{j=1}^{n-1} A_{hj}\, x^i_j$.

(We are denoting by $\{x_j\}$ etc., the $\alpha_i - h$ vector, not its j-th component. We shall sometimes write $\{x_j\}$, $j = 1, \ldots, h$, to indicate the range of j).

(b) If $A_{h+1\,h+1}$ is non-singular we put $x^i_{h+1} = A^{-1}_{h+1\,h+1}\, z^i_{h+1}$.
Then $\{x^i_j\}$, $j = 1, \ldots, h+1$, is a partial $\alpha_i - h + 1$ solution and, exactly as in 10.6, it may be shown that it satisfies $(X_i)_{h+1}$.

(c) The considerations of (c) and (d) appear necessary when $A_{h+1\,h+1}$ is singular.

Every partial $\alpha_i - h$ solution of $Ax = 0$ is expressible as

$$\{x_j\} = \kappa\{x^i_j\} + c\{\bar{x}_j\} \qquad (j = 1, \ldots, h).$$

~~Suppose $\kappa = 1$.~~

Now $\{x_j\}$ and $\{x^i_j\}$ are partial α_i -h solutions of $Ax = 0$. But $x_j = \kappa x^i_j$, $j = 1, \ldots, \min(h, \alpha_{i+1} - 1)$, by

will $\kappa \neq 0$. Hence we may suppose $\kappa = 1$ since

Theorem 2, ~~Hence as~~ $\{\bar{x}_j\}$ is also a partial $r - h$ solution of $Ax = 0$, (if $\{\bar{x}_j\} \neq 0$,) and as every such vector is an α_q-vector for some q , $\{\bar{x}_j\}$ is an $\alpha_q - h$ vector with $\alpha_i < \alpha_q \leq h$.

(d) If $\{x_j\} \supset 0$ we shall now prove $\{\bar{x}_j\} \supset 0$, $(j = 1, \ldots, h)$.
If $\{\bar{x}_j\} = 0$, the result is trivially true. Let us assume that $\{\bar{x}_j\} \neq 0$.
By (c), $\{\bar{x}_j\} = 0$, $j = 1, \ldots, \alpha_q - 1$, where $\alpha_q > \alpha_i$.
Hence we need only prove $\bar{x}_j \geqq 0$, for $\alpha_q \leq j \leq h$.

By hypothesis $R_{\alpha_p \alpha_i} = 0$, $\alpha_i < \alpha_p \leq h$.

Hence by $(X_i)_h$, $x^i_{\alpha_p} = 0$, $\alpha_i \leq \alpha_p \leq h$.

When $\{x_j\} \supset 0$, it follows that $\bar{x}_{\alpha_p} \geq 0$, $\alpha_q \leq \alpha_p \leq h$, provided that $\{\bar{x}_j\}$ is normalized so that $c \geq 0$. (Actually of course $\bar{x}_j \geq 0$). It remains to prove that $\bar{x}_j \geq 0$, when A_{jj} is non-singular.

(e) Suppose that A_{jj} is non-singular. Then $j > \alpha_i$. Let us suppose that $\bar{x}_g \geq 0$ when $\alpha_q \leq g \leq j - 1$. Then $\bar{z}_j = -\sum_{j=1}^{g-1} A_{jg} \bar{x}_g \geq 0$.

And as $A^{-1}_{jj} > 0$, $\bar{x}_j \geq 0$, follows. As we have $\bar{x}_g \geq 0$, $g = 1, \ldots, \alpha_q$ we obtain by (d) and the present section that $\{\bar{x}_j\} \geq 0$, $(j = 1, \ldots, h)$.

(f) Now suppose that $A_{h+1\,h+1}$ is singular, say $h + 1 = \alpha_m$. Let $\{x_j\} \supset 0$ $(j = 1, \ldots, h)$. We have proved that on proper normalization, $\{\bar{x}_j\} \geq 0$, and that in this case $c \geq 0$. By theorem 1, there is an x_{α_m} satisfying

$$A_{\alpha_m \alpha_m} \quad x_{\alpha_m} = z_{\alpha_m} = z^i_{\alpha_m} + c\bar{z}_{\alpha_m}, \quad (8)$$

for some $c \geq 0$, if and only if

$$z^i_{\alpha_m} + c\bar{z}_{\alpha_m} = 0 .$$

As $\bar{z}_{\alpha_m} \geq 0$, (8) can be satisfied if and only if

$$z^i_{\alpha_m} = 0 \text{, since we may put } c = 0.$$

Hence by $(Z_i)_{\alpha_m}$ there is an $x^i_{\alpha_m}$ satisfying (8) if and only if

$$R_{\alpha_m \alpha_i} = 0 .$$

If $R_{\alpha_m \alpha_i} = 0$, we put $x^i_{\alpha_m} = x_{\alpha_m} = 0$. Of course this satisfies (8), with $c = 0$, as $z^i_{\alpha_m} = 0$. The vector

$\{x^i_j\}$, $j = 1, \ldots, \alpha_m$ now satisfies $(X_i)_{\alpha_m}$.

(g) From (b) and (f) it follows that if Q_h, $\alpha_i \leq h < k$, holds so does Q_{h+1}. But there is a positive partial $\alpha_i - \alpha_{i+1} - 1$ solution of $Ax = 0$ satisfying $(X_i)_{\alpha_{i+1}-1}$, by theorem 2. This is equivalent to $Q_{\alpha_{i+1}-1}$. By induction we obtain Q_k, which is equivalent to the theorem.

<u>Corollary 1.</u> <u>Let $R_{\alpha_p \alpha_i} = 0$, $p = i+1, \ldots, \ell$, and let the α_i - vector x^i satisfy $Ax = 0$, and $(X_i)_k$. Then x^i is unique.</u>

We have $x^i_h = 0$, $j = 1, \ldots, \alpha_i - 1$, and also $A_{\alpha_i \alpha_i} x^i_{\alpha_i} = 0$. Hence $x^i_{\alpha_i}$ is unique.
If $R_{h\alpha_i} = 0$, $x^i_h = 0$.

Suppose $R_{h\alpha_i} > 0$. Then by hypothesis A_{hh} is non-singular. Let us inductively suppose that x^i_j, $j = 1, \ldots, h-1$, are unique. Then $z^i_h = -\sum_{j=1}^{h-1} A_{hj} x_j$ is also unique. Hence x^i_h is unique. But we have shown that x^i_j, $j = 1, \ldots, \alpha_i$, are unique, ~~Hence x^i_h is unique. But we have shown that x^i_j $j = 1, \ldots,$ are unique,~~ and so the Corollary follows by induction.

As Corollary 2, we may append the proposition Q_h of theorem 3, (which holds as Q_k holds), amplified by equivalent of Corollary 1.

<u>Corollary 2.</u> <u>There is a (weakly) positive partial $\alpha_i - h$ solution of $Ax = 0$, if and only if $R_{\alpha_p \alpha_i} = 0$, when $\alpha_i < \alpha_p \leq h$. When $R_{\alpha_p \alpha_i} = 0$, $\alpha_i < \alpha_p \leq h$, there is a partial α_i - h solution satisfying $(X_i)_h$. This partial</u>

solution is unique.

When $i = \ell$ the condition of theorem 3, $R_{\alpha_p \alpha_\ell} = 0$, $p = i+1, \ldots, \ell$ is trivially satisfied. We obtain Corollary 3.

Corollary 3. There is a unique positive α_ℓ-vector x^ℓ such that $Ax^\ell = 0$.

10.8.

Lemma. Let z^j $j = 1, \ldots m$, be h_j - vectors (not necessarily positive), such that $h_1 < h_2 < \ldots < h_m$. The vectors z^j, $j = 1, \ldots m$ are linearly independent.

Let us suppose that

$$z = \sum_{j=1}^{m} c_j z^j = 0.$$

Then

$$z_{h_1} = \sum_{j=1}^{m} c_j z^j_{h_1} = c_1 z^1_{h_1} \quad \text{as} \quad z^j_{h_1} = 0, \quad j \geq 2.$$

Hence $\quad c_1 = 0, \quad$ as $\quad z^1_{h_1} \neq 0.$

Repeating this argument for $j = 2, 3, \ldots, m$ successively we obtain $c_1 = c_2 = \ldots c_m = 0$.

It follows that the vectors are linearly independent.

10.9.

Let us now denote by α_{i_j}, $j = 1, \ldots, m$ $\alpha_{i_1} < \alpha_{i_2} < \ldots < \alpha_{i_m}$, those α_i for which $R_{\alpha_p \alpha_i^j} = 0$, $p = i+1, \ldots \ell$. By theorem 3 there is an α_{i_j}-solution of $Ax = 0$, satisfying $(X_{i_j})_k$ (This solution will be denoted by x^{i_j}).

Theorem 4. The vectors x^{ij}, satisfying $Ax = 0$, are linearly independent. The vector $x \supset 0$ satisfies $Ax = 0$ if and only if $x = \sum_{j=1}^{m} c_j x^{ij}$, for some constants $c_j \geq 0$.

(a) The vectors x^{ij} are α_{i_j} vectors, $\alpha_{i_1} < \alpha_{i_2} < \ldots < \alpha_{i_m}$. Hence by 10.4, Lemma the x^{ij} are linearly independent.

If $x = \sum_{j=1}^{m} c_j x^{ij}$, $c_j \geq 0$, some $c_j > 0$, then $x \supset 0$ and as $Ax^{ij} = 0$, $j = 1, \ldots, m$, it follows that $Ax = 0$.

(b) Let us suppose that $x \supset 0$ and that $Ax = 0$. By 10.5 every solution of $Ax = 0$, is an α_i -vector for some i.

Suppose that x is an α_μ - vector.

Let S_h be the set of integers such that $p \in S_h$, if and only if

(i) $\alpha_\mu \leq \alpha_p \leq h$,

(ii) $R_{\alpha_q \alpha_p} = 0$, when $\alpha_p < \alpha_q \leq h$

(It should be noted that if $\alpha_\mu = h$, $\mu \in S_h$).

If $p \in S_h$, by 10.8 Corollary 2 th theorem 3 there is a partial α_p h solution of $Ax = 0$, denoted $\{x_j^p\}$, satisfying $(X_p)_h$, and therefore also $(Y_p)_h$.

(c) Let us suppose that if $\{x_j\}$ $j = 1, \ldots, h$, $h > \alpha_\mu$, is a (weakly) positive α_μ - solution of $Ax = 0$; then there are c'_p such that

$$\{x_j\} = \sum_{p \in S_h} c'_p \{x_j^p\} \qquad c_p \geq 0, \quad p \in S_h, \quad c_\mu > 0, \qquad (9)$$

and further that there is a positive partial $\alpha_\mu - h + 1$ solution $\{x_j\}$, $j = 1, \ldots, h + 1$, whose j-th vector component equals that of (9), when $j \leq h$.

Then

$$A_{h+1\,h+1} \quad x_{h+1} = \sum_{p \in S_h} c_p' \; z_{h+1}^p \,. \tag{10}$$

(d)

If $A_{h+1\,h+1}$ is non singular we have from (10)

$$x_{h+1} = - A^{-1}_{h+1\,h+1} \left(\sum_{p \in S_h} c_p' \; z^p_{h+1} \right) \geq 0,$$

as $c_p^p \geq 0$, $z^p_{h+1} \geq 0$, $p \in S_h$ and $A^{-1}_{h+1\,h+1} > 0$.

Thus $\{x_j\}$ is a positive α_p ~~$\neq$~~ $h+1$ solution of $Ax = 0$.

(e)

Let us suppose that $h + 1 = \alpha_p$.

By theorem 1 there is an x_{h+1} satisfying (10) only if

$$\sum_{p \in S_h} c_p' \; z^p_{h+1} = 0 \,.$$

Hence as $c_p' > 0$, it follows from $(Y_p)_h$, $p \in S_h$, that

$$R_{h+1\,\alpha_p} = 0$$

~~and c_p = when p S_n~~

$$R_{h+1\,\alpha_p} = 0, \text{ when } c'_p >_n 0 \,; \quad p \in S_h \,.$$

Thus $p \in S_{h+1}$,

and $p \in S_{h+1}$ when $c_p' > 0$, $p \in S_n$.

Hence

$$\{x_j\} = \sum_{p \in S_{h+1}} c_p' \; \{x_j^p\} \qquad j = 1, \ldots, h,$$

as $x_h^p = 0$.

(f)

If these conditions are satisfied

$$x_{h+1} = c'_p \; x^p_{h+1} \,, \quad c'_p \geq 0,$$

as the vector satisfying $A_{\alpha_p \alpha_p}$ $x_{\alpha_p} = 0$ is unique and also positive when $(X_p)_{\alpha_p}$ holds. Further, by $(X_p)_{h+1}$

$$x^p_{h+1} = 0, \quad p \in S_{h+1}\,, \quad p \neq h+1 \,.$$

Hence $\quad \{x_j\} = {\sum_{p \in S_{h+1}}}' c_p' \{x_j^p\} \quad j = 1, \dots, h+1.$

This result we have already proved when $A_{h+1\ h+1}$ is non-singular.

(9)
But evidently $\quad \{x_j\} = c_\mu' \{x_j^\mu\} \quad j = 1, \dots, \alpha_\mu$

and S_{α_μ} consists of μ only.

After a trivial renumbering of the c_p', we obtain the theorem by induction.

We have proved that every positive latent column vector of A associated with 0 is a linear combination of a set of positive latent column vectors. A similar result naturally holds for row vectors. It might also equally well be enunciated in terms of latent column or row vectors of P associated with ρ. It is worth emphasising that we have not proved that all latent column or row vectors of A associated with 0 are linear combinations of a set of positive latent vectors. This is not true in general.

10.10.

In this section we shall return to the matrix $P = \rho I - A$. The following theorem is due to Frobenius (1912). : "There is a latent column vector $x > 0$ of P associated with λ, a latent root of P, if and only if there is an arrangement $(h_1, h_2 \dots h_k)$ of $(1, 2, \dots k)$ such that

(i) The matrix $\bar{P}$ is in normal form, where

$$\bar{P} = [\bar{P}_{ij}] \qquad i, j = 1, \dots, k$$

and $\quad \bar{P}_{h_\mu h_\nu} = P_{\mu\nu}.$

(ii)

There is an r, such that $\lambda = \rho_r$, the non-negative greatest latent root of P_{rr},

and (iii) $\rho_s < \rho_r$ when $h_s > h_r$.

The latent vector $\bar{x}$ of $\bar{P}'$ obtained by Frobenius is in our terminology an h_r- solution of $(\bar{P} - \rho_r I)\bar{x} = 0$.

From (ii) we immediately have

$$x_s = \bar{x}_{h_s} = 0 \qquad (11)$$

when $\rho_s \geq \rho_r \quad (s \neq r)$,

while by 7.4 it is easily proved that

$$x_r = \bar{x}_{h_r} > 0. \qquad (12)$$

(We are here denoting by x the vector for which $x_s = \bar{x}_{h_s} \quad s = 1, \ldots, k.$)

Clearly $(P - \rho_r I)\ x = 0$;

We are now concerned only with the case $\lambda = \rho$, i.e. $r = \alpha_i$ for some i, $1 \leq i \leq b$.

Let $r = \alpha_i$, and denote the vector satisfying the theorem of Frobenius by x^i.

From (11) it follows that

$$x^i_{\alpha_j} = 0, \quad j \neq i \qquad (13)$$

and by (12) $x^i_{\alpha_i} > 0$. $\qquad$ (14)

By some other arrangement of $(1, 2, \ldots, k)$ we may similarly obtain the vectors, $x^{i_1}, x^{i_2}, \ldots, x^{i_m}$.

By 10.8 Lemma it follows from (13) and (14) that these vectors are linearly independent, and hence we may in this case modify Frobenius' theorem:

"If there are m permutations $(h_1^j, h_2^j, \ldots, h_k^j)$, $j = 1, \ldots, m$, of the integers $(1, 2, \ldots, k)$ so that $h^j_{\alpha_{i_j}} > h^j_{\alpha_s}$, $s \neq i_j$, and $i_j \neq i_{j'}$, when $j \neq j'$, and there is no permutation so that $h_{\alpha_t} > h_{\alpha_s}$, $t \neq s$, when $s \neq i_j$ for some j, $1 \leq j \leq m$, then there are precisely m linearly independent latent column vectors associated with ρ."

The equivalence of this theorem and theorem 3 (i), together with the first part of theorem 4, follows from 3, 6, Theorem 4, provided that the x^i of this section are those of section 10.7 and 10.9 .

That the $x^{i_j}_j$ of the previous section are indeed those of the present one is easily proved. Let us now denote the vector x constructed above by $\hat{x}^{i_j}$, reserving x^{i_j} for the vectors of Theorem 4.

By Theorem 4,

$$\hat{x}^{i_q} = \sum_{j=1}^{m} c_j \, x^{i_j}_{(k)} .$$

As $x^{i_j}_{\alpha_p} = 0$, $\alpha_p \neq \alpha_{i_j}$ and $x_{\alpha_{i_j}} > 0$, we have from (11) that

$$c_j = 0, \quad j \neq q ,$$

and hence

$$\hat{x}^{i_q} = c_q \, x^{i_q} .$$

10.11.

We have so far confined ourselves to the case of latent column vectors associated with 0. The results for latent row vectors follow easily. The transposed matrix A' is not in general in normal form. If however we reverse the order of the indices (1, 2, .., k) and carry out the corresponding conjugate permutation on A to obtain $\bar{A}$, then $\bar{A}'$ is in normal form, for

$$\bar{A}'_{ij} = \bar{A}_{ji} = A_{k+1-j\; k+1-i} = 0,$$

when $i < j$.

Hence there is an $k+1-\alpha_i$ solution of $\bar{A}'\bar{y} = 0$ if and only if $R_{\alpha_i \alpha_p} = \bar{R}'_{k+1-\alpha_p\; k+1-\alpha_i} = 0$ for $p = 1, 2, \ldots, i-1$.

This follows by theorem 3.

(The $\bar{R}'_{ij}$ are supposed defined for $\bar{A}'$ as R_{ij} for A. It is easily seen that $R_{ij} = \bar{R}'_{k+1-j\; k+1-i}$.)

Here we have a row vector y ($y_j = \bar{y}_{k+1-j}$) satisfying

$$y'A = 0, \tag{15}$$

and

$$\left.\begin{aligned} y_i &= 0 \quad \text{when} \quad i = \alpha_i+1, \alpha_i+2, \ldots, k \\ y_{\alpha_i} &\neq 0 . \end{aligned}\right\} \tag{16}$$

We shall call a row vector satisfying (15) and (16) an α_i-solution of $y'A = 0$. A vector satisfying (16) will be called an α_i-vector.

We shall state the analogues of theorems 3 and 4.

<u>Theorem 3a.</u> <u>There is a (weakly) positive α_i- solution of $y'A = 0$ if and only if</u>

$$\underline{R_{\alpha_i \alpha_p} = 0 \qquad \text{for } p = 1, 2, \ldots, i-1.}$$

(ii) <u>If $R_{\alpha_i \alpha_p} = 0$, $p = 1, \ldots. i-1$, there is a α_i-vector y^{i} satisfying $y'A = 0$, such that</u>

$(Y_i)_1$: $y^i_j > 0$ when $R_{\alpha_i j} > 0$,

$y^i_j = 0$ when $R_{\alpha_i j} = 0$,

for $j = 1, 2, \ldots, \alpha_i - 1, \alpha_i$.

Corollary 1. Let $R_{\alpha_i \alpha_p} = 0$, $p = 1, \ldots, i-1$, and let the vector $y^{i\prime}$ satisfy $y^{i\prime}A = 0$ and $(Y_i)_1$. Then y^i is unique.

Corollary 2. There is a unique α_1-vector $y^{1\prime}$ satisfying $y^{1\prime}A = 0$.

Theorem 4a. Let α_{i_j} $j = 1, \ldots, \mu$, denote all those α_i for which $R_{\alpha_i \alpha_p} = 0$, $p = 1, \ldots, i-1$. The vectors $y^{i_j\prime}$, satisfying $y'A = 0$, whose existence is guaranteed by theorem 3a, are linearly independent. The vector $y' > 0$ satisfies $y'A = 0$ if and only if $y = \sum_{j=1}^{\mu} c_j \; y^{i_j}$ for some constants $c_j \geq 0$.

It should be noted that in general the number of positive latent column vectors of A associated with 0 need not equal the number of similar latent row vectors. Examples of this will appear later, Cf. 10.24.

10.12.

Let A be a matrix for which

$$R_{\alpha_j \alpha_i} = 0 \qquad i \neq j, \quad i,j = 1, \ldots, \ell \qquad (16)$$

By theorems 4 and 4a A has ℓ linearly independent positive latent column vector x^i and latent row vectors $y^{i\prime}$, $i = 1, \ldots, \ell$ associated with 0. The latent roots of A are those of the A_{ii} and by 8.8. (with $P = \xi I + A$) the latent root 0 is a single latent root of $A_{\alpha_i \alpha_i}$ $i = 1, \ldots, \ell$. Hence the multiplicity of the root 0 equals the number of latent (column) vectors associated with it, and by 6.3, Theorem 1, this implies that the classical canonical submatrices associated with 0 are of order 1.

We may say rather more.

As $R_{\alpha_j \alpha_i} = 0 \quad (i \neq j)$ implies

$$R_{\alpha_j s} \cdot R_{s\alpha_i} = 0 \qquad (s = 1, 2, \dots, k)$$

It follows from theorems 3 and 3a

that $\underset{\smile}{y_s^{j\prime}} \; x_s^i = 0 \qquad (s = 1, 2, \dots, k), \; i \neq j,$

and hence

$$\underset{\smile}{y^{j\prime}} \; x^i = \sum_{s=1}^{k} y_s^{j\prime} \; x_s^i = 0 \quad , \; i \neq j. \qquad (17)$$

Vectors y, x satisfying $y'x = x'y = 0$ are called orthogonal.

Further $\quad y_{\alpha_i}^{i\prime} \; x_{\alpha_i}^i > 0$

and $\quad y_s^{i\prime} \; x_s^i = 0 \quad ; \quad s \neq \alpha_i$, as $y_s^i = 0, \; s > \alpha_i$,

whence $\quad y^{i\prime} \; x^i = \sum_{s=1}^{k} y_s^{i\prime} \; x_s^i > 0, \quad \underline{x}_s^i = 0, \; s < \alpha_i.$

We have left the vectors x^i, y^i undefined in respect of a positive scalar factor. We may choose x^i, y^i so that

$$\underset{\smile}{y^{i\prime}} \; x^i = 1. \qquad (18)$$

The process of multiplying y^i, x^i, by a factor so that (18) is satisfied is called "normalization". When the factor is positive, we shall call it "positive normalization".

A vector $x \geq 0$ $(x > 0)$ remains positive (strictly positive) on positive normalization.

Two sets of vectors x^i, y^i, $i = 1, \dots, l$ satisfying

(17) and (18) are called biorthonormal sets of vectors.
We have the theorem:

Theorem 5. If $R_{\alpha_j \alpha_i} = 0$ $i, j = 1, \ldots, l$, $i \neq j$ the classical canonical submatrices of $A(P)$ associated with $0(\rho)$, are of order 1. There exist bi-orthonormal sets of positive latent column and row vectors associated with $0(\rho)$. (The results for P are of course obtained by considering $\rho I - A$.)

10.13.

Let us now put

$$Q_1 = [x^1, x^2 \ldots\ldots x^l],$$
$$U_1' = [y^1, y^2 \ldots\ldots y^l]$$

and let $E_1 = Q_1 U_1$.

Comparing with 6.9, we see that E_1 is the principal idempotent element of A associated with 0. It follows that

$$E_1 = \sum_{i=1}^{l} x^i y^{i\prime} > 0 .$$

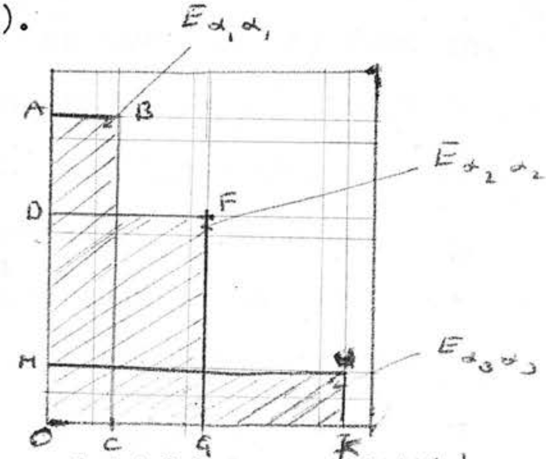
The classical canonical submatrices are of order 1, and so $N_1 = 0$.

Theorem 6. If $R_{\alpha_j \alpha_i} = 0$, $i, j = 1, \ldots, l$ $i \neq j$, the principal idempotent element of $A(P)$ associated with $0(\rho)$ is (weakly) positive.

~~As E is a function of A,~~ The matrix E_1 is not strictly positive unless A is irreducible (in which case $E = x' y'' > 0$). ~~Cf. 6.5.~~ The situation is best illustrated diagrammatically.

(Here $\ell = 3$).

$E = E_1 =$



The three shaded areas (OABC, ODFG, OHJK) correspond to $x^1\, y^{1\prime}$, $x^2\, y^{2\prime}$, $x^3\, y^{3\prime}$ respectively. Elements in the unshaded area are certainly zero. More generally, if $E = E_1$ is partitioned conformally with A,

$$E_{rs} = \sum_{i=1}^{\ell} x_r^i \, y_s^{i\prime} ,$$

whence $E_{rs} \neq 0$ if and only if there is an i such that

$$R_{r\alpha_i} > 0 \quad \text{and} \quad R_{\alpha_i s} > 0 \qquad (\text{by } (X_i)_k, (Y_i)_1).$$

In particular it follows that,

$$E_{rs} = 0 , \quad \text{if} \quad \alpha_j \leq r < \alpha_{j+1} \text{ and } \alpha_j < s ,$$
$$j = 0, 1, \ldots, \ell, \cancel{+1}$$

where we conventionally put $\alpha_0 = 0$, $\alpha_{\ell+1} = k + 1$.

10.14.

The R-conditions are in the nature of "nought and cross" conditions. We mean by that, that it is sufficient to know whether $A_{ij} \neq 0$ or $A_{ij} = 0$ to determine them completely; the values of the non-zero elements are immaterial. It is natural to enquire whether it is possible to determine the classical canonical submatrices of A associated with 0 from such "nought and cross" conditions. In one special case, that of 10.12 we have already done so. In general, however, it is not possible to do so, though considerable progress can be made in that direction. In particular, if l, the number of singular A_{ii}, does not exceed three, the classical canonical submatrices of A associated with 0 may be completely determined from R-conditions.

We shall also be concerned with the principal idempotent and nil-potent elements associated with 0. In view of the preceding results it is natural to enquire whether the generalized latent vectors and idempotent elements, etc., may be chosen positive. It appears preferable to attempt to answer this question for the positive matrix $P = \rho I - A$, and its latent root ρ. The generalized latent vectors here satisfy equations of the type (in the notation of 6.2)

$$P x_r = \rho x_r + x_{r-1} \qquad r \geq 2,$$

$$P x_1 = \rho x_1,$$

and thus reverting to the matrix A,

$$\left.\begin{aligned} A x_r &= (\rho I - P) x_r = -x_{r-1}, \; r \geq 2, \\ A x_1 &= (\rho I - P) x_1 = 0 . \end{aligned}\right\} \quad (19)$$

Where generalized latent row vectors are concerned we shall similarly be considering equations of the type

$$\left.\begin{aligned} y_r' A &= y_r' (\rho I - P) = -y_{r+1}, \; r < \nu_{ij}, \\ y_r' A &= y_r' (\rho I - P) = 0, \; r = \nu_{ij} . \end{aligned}\right\} \quad (20)$$

As it will be convenient to denote α_i-vectors by $x^i (y^i)$ and their j-th vector component by x^i_j (y^i_j), we shall not use the notation of Chapter 6. (When x^i is partitioned, $\{x^i_j\}$, $j = 1, \ldots, k$, conformably with A then x^i_j is the j-th vector component of x^i.)

We shall first prove some theorems on positive generalized latent vectors. Then in 10, we shall consider the cases (= 1, 2, 3).

10.15.

We shall be concerned with the positivity of the generalized latent vectors. In this connection, we shall prove a theorem, theorem 9. First we prove two preliminary theorems, theorem 7 being of a trivial nature. These theorems are not actually required for the proof of theorem 9, but they shed some light on it.

Theorem 7. Let x be an h - vector, and $w = A x$, a g-vector.
Then $g \geq h$, and if A_{hh} is non-singular, $g = h$.

For $\quad w_i = \sum_{j=1} A_{ij} x_j, \quad i = 1, \ldots, k$

Hence $\quad w_i = 0, \quad i < h,$

whence $\quad g \geq h$.

We have $\quad w_h = A_{hh} x_h,$

where $x_h \neq 0$. If A_{hh} is non-singular, it follows that $w_h \neq 0$.

Hence $g = h$, in this case;

Theorem 8. Let $w > 0$ be an α_j-vector, and $A x = - w$.
Then x is an α_j- vector, $\alpha_j < \alpha_i$.

Let x be an h - vector.

By theorem 7, $h \leq \alpha_i$. If $h = \alpha_i$,

$$A_{\alpha_i \alpha_i} \quad x_{\alpha_i} = w_{\alpha_i} > 0$$

by hypothesis, and this is impossible by theorem 1.

Hence $h < \alpha_i$

The second part of theorem 7 implies $h = \alpha_j$ for some j.

Corollary. *If $A x = - w$ and $w > 0$ is an α_i-vector, then $\alpha_i > \alpha_1$.*

10.16.

Let $\mathbf{A}' = (\alpha_1' \dots \alpha_p')$, $\mathbf{A}'' = (\alpha_1'', \alpha_2'' \dots \alpha_q'')$ be subsets of $\mathbf{A} = (\alpha_1, \alpha_2 \dots \alpha_\ell)$, where

$$\mathbf{A}' \cup \mathbf{A}'' = \mathbf{A},$$

$$\mathbf{A}' \cap \mathbf{A}'' = 0 .$$

We shall suppose $\mathbf{A}$, $\mathbf{A}'$, $\mathbf{A}''$ ordered so that, $\alpha_i < \alpha_{i+1}$,
$\alpha_i' < \alpha_{i+1}'$, $\alpha_i'' < \alpha_{i+1}''$.

Theorem 9. *If $R_{\alpha_g'' \alpha_i'} = 0$, $g = 1, \dots, q$, $i = 1 \dots, p$*

and $R_{\alpha_g' \alpha_i'} > 0$, $g > i$, $i, g = 1, \dots, p$

then there is a weakly positive sequence of generalized latent vectors x^i, $i = 1, \dots, p$ such that

$$A x^i = - x^{i+1}, \quad i = 1, \dots, p - 1,$$

$$A x^p = 0$$

satisfying $\overline{X}^i$, where

$\overline{X}^i$:

$$x_h^i > 0 \quad \text{if} \quad \sum_{g=i}^{p} R_{h \alpha_g'} > 0,$$

$$x_h^i = 0 \quad \text{if} \quad \sum_{g=i}^{p} R_{h \alpha_g'} = 0,$$

~~We may note~~ for $h = 1, 2 \dots k$.

We may note :

(1) We have slightly changed our notation. By x^i we now denote an α_i'-vector, not an α_i-vector as before;

(2) The sequence x^i, $i = 1, \ldots, p$ is written in an order opposite to that of Chapter 6.

Proof.

(a) If $\alpha_j > \alpha_p'$ then $\alpha_j \in \mathcal{A}''$.
As $R_{\alpha_f \alpha_p'} = 0$ if $\alpha_f < \alpha_p$, the condition $R_{\alpha_j'' \alpha_p'} = 0$ is equivalent to $R_{\alpha_f \alpha_p'} = 0$ when $\alpha_f \neq \alpha_p$. The condition $R_{\alpha_g' \alpha_i'} > 0$ when $g > i$ does not apply when $i = p$.

(b) When $i = p$ the restatement of the theorem is equivalent (by (a)) to : "If $R_{\alpha_g \alpha_p'} = 0$, $\alpha_g \neq \alpha_p'$, there is a (weakly) positive primary latent α_p'-vector x^p such that

$$x_h^p > 0 \quad \text{if} \quad R_{h\alpha_p'} > 0,$$

$$x_h^p = 0 \quad \text{if} \quad R_{h\alpha_p'} = 0, \text{ for } h = 1, 2, \ldots, K.$$

This is theorem 3; hence there is an x^p satisfying the conditions of the theorem.

(c) Let us assume there is a set S_{s+1}^* of generalized latent vectors x^i, $i = s+1, \ldots, p$, x^i satisfying $\overline{X}^i$.
(We do not need to postulate that x^i is an α_i'-vector, this is ensured by $\overline{X}^i$).
Let S_{s+1}' be the set of vectors

$$w^i = x^i + a_1 x^{i+1} + a_2 x^{i+2} + \ldots + a_t x^{i+t},$$

$$i = s+1, \ldots, p \text{ and } t \leq p - s - 1.$$

We are here adopting the convention

$$x^i = w^i = 0 \quad \text{when} \quad i > p.$$

Then S'_{s+1} is a set of generalized latent vectors of A and there is a constant γ such that w^i, $i = s+1, \ldots, p$, satisfies $\overline{X}^i$ when $a_1 \geq \gamma$.

We have

$$A w^i = \sum_{r=0}^{t} A a_r x^{i+r} = -\sum_{r=0}^{t} a_r x^{i+r+1} = -w^{i+1}$$
$$i = s+1, \ldots, p,$$

where $a_0 = 1$.

This proves the first part of the statement.

Suppose that $\sum_{g=i+1}^{p} R_{h\alpha'_g} = 0$,

Then $\sum_{g=m}^{p} R_{h\alpha'_g} = 0$ when $m \geq i + 1$.

Hence $x_h^{i+1} = x_h^{i+2} = \ldots = x_h^p = 0$,

and so $w_h^i = x_h^i$.

It follows by $\overline{X}^i$ that

$$w_h^i > 0 \quad \text{when} \quad \sum_{g=i}^{p} R_{h\alpha'_g} > 0,$$
$$w_h^i = 0 \quad \text{when} \quad \sum_{g=i}^{p} R_{h\alpha'_g} = 0,$$

for all a_1.

Suppose that $\sum_{g=i+1}^{p} R_{h\alpha'_g} > 0$.

Then $\sum_{g=i}^{p} R_{h\alpha'_g} > 0$.

Now $x_h^{i+1} > 0$, by $\overline{X}^{i+1}$.

But $w_h^i = x_h^i + a_1 x_h^{i+1} + \ldots + a_t x_h^{i+t}$,

and so $w_h^i > 0$ when $a_1 \geq \gamma_h^i$, say.

Let $\gamma_j^i = 0$ when $\sum_{g=i+1}^{p} R_{j\alpha'_g} = 0$,

and put $\gamma = \max \gamma_j^i$ $j = 1, \ldots, k$, $i = s+1, \ldots, p$.

Then $w_h^i > 0$ when $a_1 > \gamma$, and we have proved

$\sum_{g=c}^{p} R_{h\alpha'_g} > 0$.

Hence w^i satisfies $\overline{X}^i$ when $a_1 > \gamma$, $i = s+1, \ldots, p$.

(d) Let $\{x_j^s\}$ be a partial α'_s - h vector, and let $\{x_j^s\}$ satisfy $\overline{X}_h^s$, where

$$\overline{X}_h^s \quad : \quad x_j^s > 0 \quad \text{if} \quad \sum_{g=s}^{p} R_{h\alpha'_g} > 0 ,$$

$$x_j^s = 0 \quad \text{if} \quad \sum_{g=s}^{p} R_{h\alpha'_g} = 0$$

$$\text{when} \quad j = 1, \ldots, h .$$

By an argument almost identical with that of (c) we see that

$$\{w_j^s\} = \{x_j^s + \beta\, x_j^{s+1}\}, \quad j = 1, \ldots, h, \quad \beta \geq 0$$

also satisfies $\overline{X}_h^s$.

(e) Let us assume that there is a set S_{s+1} of generalized latent vectors, x^i, $i = s+1, \ldots, p$, where x^i satisfies $\overline{X}^i$.

Let P_h be the proposition:

P_h : There is a set S'_{s+1} of generalized latent vectors w^i, $i = s+1, \ldots, p$, such that w^i satisfies $\overline{X}^i$, and in addition there is a partial α'_s - h solution $\{w_i^s\}$ of $A\, w = - w^{s+1}$ such that $\{w_j^s\}$ satisfies $\overline{X}_h^s$.

(f) Let x_j^s be the partial $\alpha'_s - \alpha'_s$ solution of $A\, x = 0$ such that

$$x_j^s = 0 , \; j = 1, \ldots, \alpha'_s - 1$$

and $x_{\alpha'_s}^s$ is the unique strictly positive vector ~~x~~ satisfying

$$A_{\alpha'_s \alpha'_s} \; x_{\alpha'_s}^s = 0 .$$

As $x_j^{s+1} = 0 \quad j \leq \alpha'_s$ (since $\alpha'_{s+1} > \alpha'_s$) and $\sum_{g=s}^{p} R_{h\alpha'_g} = 0 \quad j \;\alpha'_s$, while $\sum_{g=s}^{p} R_{h\alpha'_g} = 1$, it follows immediately that $\{x_j^s\}$ satisfies $\bar{X}^s_{\alpha'_s}$, and that ~~Hence~~ $P_{\alpha'_s}$ is satisfied for $w^i = x^i$, $i = s+1, \ldots, p$ and $\{w_j^s\} = \{x_j^s\}$ $\quad j = 1, \ldots, \alpha'_s$.

(g) Let us assume P_{h-1} and construct vectors satisfying P_h. $h-1 \geq \alpha'_s$
The argument is very similar to that of theorem 3.

Let x^i, $i = s+1, \ldots, p$ and $\{x_j^s\}$, $j = 1, \ldots, h-1$ be vectors satisfying P_{h-1}.

We put $z_h^s = - \sum_{j=1}^{h-1} A_{hj} \; x_j^s$,

As $A_{hj} \leq 0$, and $x_j^s \geqq 0$, it follows that $z_j^s \supseteq 0$; Cf. 4.25, 4.26.

But $z_h^s > 0$, if and only if there is a j, $1 \leq j \leq h-1$, such that $A_{hj} < 0$ and $x_s^j > 0$.

By definition of r_{hj} and by $\bar{X}^s_{h-1}$ this is equivalent to : "$z_h^s > 0$ if and only if there is a j, $1 \leq j \leq h-1$, for which $r_{hj} > 0$ and $\sum_{g=s}^{p} R_{j\alpha'_g} > 0$."

Since $\sum_{g=s}^{p} R_{h\alpha'_g} \geq \sum_{j=1}^{h} (r_{hj} \sum_{g=s}^{p} R_{j\alpha'_g})$

and either $\sum_{g=s}^{p} R_{h\alpha'_g} > 0$ or $\sum_{g=s}^{p} R_{h\alpha'_g} = 0$, we deduce that

$$z_h^s > 0 \qquad \text{if} \qquad \sum_{g=s}^{p} R_{h\alpha'_g} > 0,$$

$$z_h^s = 0 \qquad \text{if} \qquad \sum_{g=s}^{p} R_{h\alpha'_g} = 0.$$

(h) When $w^i = x^i$, $i = s+1, \ldots, p$, and $\{x_j^i\}$ $j = 1, \ldots, h$, satisfies P_h we have

(F) $\quad A_{hh} \; x_h^s = - x_h^{s+1} + z_h^s$.

Suppose (α) $\sum_{g=s}^{p} R_{h\alpha'_g} = 0$.

Then also $\sum_{g=s+1}^{p} R_{h\alpha'_g} = 0$.

By (g) $z_h^s = 0$, and by $\overline{X}^{s+1}$, $x_h^{s+1} = 0$.

Hence (F) reduces to

(F_1) $\quad A_{hh} \; x_h^s = 0$.

Thus $x_h^s = 0$ satisfies (F) and putting

$\{w_j^s\} = \{x_j^s\}$ $j=1,\ldots, h$ and $w^i = x^i$, $i = s+1,\ldots,p$,

we have found vectors satisfying P_h

when $\sum_{g=s}^{p} R_{h\alpha'_g} = 0$.

(i) Let us assume (β) : (β_1) $\sum_{g=s+1}^{p} R_{h\alpha'_g} = 0$, but

(β_2) $\sum_{g=s}^{p} R_{h\alpha'_g} > 0$.

It follows that $R_{h\alpha'_s} > 0$, $R_{h\alpha'_{s+1}} = 0$.

As by hypothesis $R_{\alpha''_t\alpha'_s} = 0$, we have $h \neq \alpha''_t$, $t=1, \ldots, q$.

Also by hypothesis $R_{\alpha'_t\alpha'_{s+1}} > 0$ when $t \geq s+1$, and

as $h \geq \alpha'_s$, we see that $h \neq \alpha'_t$ $t = 1, \ldots, p$.

Hence A_{hh} is non-singular.

By (β_2) and (g), $z_h^s \supset 0$,

and by (β_1) and $\overline{X}^i$, $x_h^{s+1} = 0$.

~~Hence~~ Hence (F) reduces to

(F_2) $A_{hh} \; x_h^s = z_h^s \supset 0$.

But $A_{hh}^{-1} > 0$, and hence $x_h^s > 0$.

It follows that $\{w_j^s\} = \{x_j^s\}$ $j = 1, \ldots, h$ and

$w^i = x^i$, $i = s+1, \ldots, p$ satisfy P_h .

(j) Finally let us assume (γ) : $\sum_{g=s+1}^{p} R_{h\alpha'_g} > 0$.
This is the most troublesome case. We shall find it necessary to subdivide it further.

The assumption (γ) implies that $\sum_{g=s}^{p} R_{h\alpha'_g} > 0$.
As in (i) we obtain $h \neq \alpha''_t, t = 1, \ldots, q$.
By (γ) and $\overline{X}^i$, $x_h^{s+1} > 0$,
and by (g) $z_h^s > 0$.
If A_{hh} is non-singular we obtain an x_h^s from (F) .
Consideration of P_h , may be postponed to (ℓ) .

(k) Let us suppose that A_{hh} is singular. We have proved that $h \neq \alpha''_t$, $t = 1, \ldots, q$.
Hence $h = \alpha'_t$, for some t.
But $h > \alpha'_s$, whence $t \geq s + 1$.

Suppose that $t = s + 1$.
Instead of (F) we may consider

$$(F_3) \qquad A_{\alpha'_{s+1}\alpha'_{s+1}} \; x^s_{\alpha'_{s+1}} = - c \; x^{s+1}_{\alpha'_{s+1}} + z^s_{\alpha'_{s+1}} .$$

Here $A_{\alpha'_{s+1}\alpha'_{s+1}}$ is singular, adj $A_{\alpha'_{s+1}\alpha'_{s+1}} > 0$,
$x^{s+1}_{\alpha'_{s+1}} > 0$, and $z^s_{\alpha'_{s+1}} > 0$. By theorem 12, Corollary 2
it follows that there is an $x^s_{\alpha'_{s+1}}$ and a positive c satisfying (F_3) .
Let us put $\{w^s_j\} = \{x^s_j\} / c$, $j = 1, \ldots, \alpha'_{s+1}$.
It is obvious that $\{w^s_j\}$ = $j = 1, \ldots, \alpha'_{s+1} - 1$ satisfies $\overline{X}^s_{\alpha'_{s+1}}$,
and as $x^{s+1}_j = 0$, $j = 1, \ldots, \alpha'_{s+1} - 1$,
$\{w^s_j\}$, $j = 1, \ldots, \alpha'_{s+1}$ is a partial $\alpha'_s - \alpha'_{s+1}$ solution

of $A w = - x^{s+1}$.

(ℓ) Let us now consider P_h in the case (γ) and A_{hh} non-singular, or $h = \alpha'_{s+1}$.

By (j) or (k) there is a vector $\{x^s_j\}$, $j = 1, \ldots, h$, where $\{x^s_j\}$, $j = 1, \ldots, h-1$ satisfies $\overline{X}^s_{h-1}$ and $x^{s+1}_h > 0$.

Hence there is a $\beta \geq 0$ such that $x^s_h + \beta x^{s+1}_h > 0$, and by (d) and ($\gamma$)

$$\{w^s_{hj}\} = \{x^s_j + \beta x^{s+1}_j\}$$

satisfies $\overline{X}^s_h$.

Let $w^i = x^i + \beta x^{i+1}$, $i = 1, \ldots, p$.

By (c) the vectors w^i are a set of generalized latent vectors w^i satisfying $\overline{X}^i$, $i = s+1, \ldots, p$.

Also $\{w^s_j\}$ is a partial $\alpha'_s - h$ solution of

$$A w = - w^{s+1}.$$

Hence we have constructed a set of vectors, w^i, $i = s+1, \ldots, p$ and a vector $\{w^s_j\}$, $j = 1, \ldots, h$ satisfying P_h.

(m) Let us suppose that (γ) holds and that $h = \alpha'_t$, $t > s + 1$.

By $\overline{X}^t$, $x^t_{\alpha'_t} > 0$.

Hence by theorem 1, Corollary 1, there is a vector x^s and a c such that

$$A_{\alpha'_t \alpha'_t} x^s_{\alpha'_t} = - x^{s+1}_{\alpha'_t} - c\, x^t_{\alpha'_t} + z^s_{\alpha'_t}.$$

As in (ℓ) $\{w^s_j\} = \{x^s_j + \beta x^{s+1}_j\}$, $j = 1, \ldots, h = \alpha'_t$, satisfies $\overline{X}^s_h$, when $\beta \geq \beta'$, say.

By (c) there is a γ such that

$$w^i = x^i + \beta x^{i+1} + c x^{i+t-s-1}, \quad i = s+1, \ldots, p,$$

satisfies $\bar{X}^i$, if $\beta \geq \gamma$.

Now $\{w_j^s\}$ $j = 1, \ldots, \alpha'_t$ is an $\alpha'_s - \alpha'_t$ solution of $A w = - w^{s+1}$, as $x^t_{j\,t} = 0$ when $j < \alpha'_t$. Hence if $\beta \geq \max (\beta'_s, \gamma)$ the vectors w^i, $i = s+1, \ldots, p$ and the vector $\{w_j^s\}$, $j = 1, \ldots, \alpha'_t$ satisfy P_h. This concludes the case (γ).

(n) The cases (α), (β), (γ), are (exclusive and) exhaustive. Hence assuming P_{h-1} we have constructed a set of vectors satisfying P_h. But in (f) we proved $P_{\alpha'_s}$, and therefore we may obtain P_k by induction. However P_k is equivalent to the assertion that there is a set S'_s of generalized latent vectors w^i so that

$$A w^i = - w^{i+1} \quad (w^{p+1} = 0), \quad i = s, \ldots, p,$$

where w^i satisfies $\bar{X}^i$, provided that there is a similar set S_{s+1} of vectors x^i, $i = s+1, \ldots, p$, where x^i satisfies $\bar{X}^i$.

(o) The set S_p consists of the single primary latent vector x^p satisfying $\bar{X}^p$. The existence of this vector was proved in (b). The existence of a set S_1 satisfying the conditions of the theorem follows by induction on s, and this is equivalent to the theorem itself.

We have proved the theorem.

10.17.

The theorem corresponding to theorem 9 in the case of row vectors is

Theorem 9a.

If $R_{\alpha'_i \alpha''_g} = 0$ when $i = 1, \ldots, p;\ g = 1, \ldots, q,$

and $R_{\alpha'_i \alpha'_g} \geq 0$ when $i, g, = 1, \ldots, p; \quad g < i,$

then there is a sequence of weakly positive generalized latent row α'_i-vectors y^i, $i = 1, \ldots, p$ such that

$$y^{i'} A = - y^{i-1} \qquad i = 2, \ldots, p\,,$$

$$y^{1'} A = 0 \quad ,$$

y^i satisfying $\overline{Y}^i$ where

$$\overline{Y}^i : \quad y^i_h > 0 \qquad \text{when} \qquad \sum_{g=1}^{i} R_{\alpha'_g h} > 0\,,$$

$$y^i_h = 0 \qquad \text{when} \qquad \sum_{g=1}^{i} R_{\alpha'_g h} = 0\,.$$

This follows from theorem 9 by considerations similar to those of 10.11.

It should be noted that the vectors of theorem 9 (or 9a) are not always part of a complete set of generalized latent vectors; Cf. 6.8. Whether they are or are not depends on the values of the $R_{\alpha'_i \alpha''_g}$ in the case of column vectors, on the values of the $R_{\alpha''_g \alpha'_i}$ in the case of row vectors. A theorem may be proved in this connection, but we can not do so here.

10.18.

In the next sections we shall consider the connection between the conditions of the matrix A and the orders of the classical canonical submatrices. The investigation will be pursued by means of the generalized latent vectors of A, as defined in 10.14. Before we consider the general case, we shall study this problem, when the number of singular A, (and therefore the multiplicity of the latent root 0) , is 1, 2, and 3.

10.19.

The case $l = 1$.

This is a trivial case, and is a particular case of that of 10.12. There is a latent column α_1 vector $x^1 > 0$, and a latent row α_1-vector $y^1 > 0$, associated with the latent root 0. The multiplicity of 0 is one. Hence there is one classical canonical submatrix of order 1 associated with 0.

10.20.

The case $l = 2$.

We may divide this case into two "types" .

(1) $R_{\alpha_2\alpha_1} = 0$, (2) $R_{\alpha_2\alpha_1} > 0$.

There is no other possibility.

(1) This again is a particular case of 10.12. There are two primary latent column vectors x^1 , x^2 , and two primary latent row vectors y^1 , y^2 , We may chose x^i $(y^{i'})$ to be a positive α_i-vector, and the sets x^1, x^2 and y^1, y^2 to be biorthonormal; cf. 10.12, Theorem 5. It also follows from that theorem that there are two classical canonical submatrices of order 1 associated with 0.

(2) $\qquad R_{\alpha_2\alpha_1} > 0$.

Theorem 9 applies with $\mathcal{A} = \mathcal{A}' = (\alpha_1, \alpha_2)$.
Hence there is a primary latent α_2 -vector x^2 , and a secondary α_1 -vector x^1 , both of which may be chosen positive. Similarly, from theorem 9a it follows that there is a primary latent row α_1- vector $y^{2\prime}$, and a secondary α_2 -vector $y^{1\prime}$, both of which may be chosen positive. The multiplicity of 0 is two. Hence by 6.7. there is one classical canonical submatrix of order two associated with 0.

That there is a classical canonical submatrix of order two associated with 0 may also be proved otherwise. For suppose not, then there are two classical canonical submatrices of order one associated with 0, and therefore also two primary latent column vectors associated with 0.

By theorem 3, Corollary 1, there is a unique latent α_2 -vector, x^2. As every solution of $A\,x = 0$ is an α_i-vector, the other latent vector must be an α_1 -vector, say x^1 .

The partial $\alpha_1 - \alpha_2 - 1$ vector $\{x^1_j\}$, $j = 1, \ldots, \alpha_2 - 1$ is a partial $\alpha_1 - \alpha_2 - 1$ solution of $A\,x = 0$. Hence by 10.6 , theorem 2 , it is the unique vector satisfying $(X_i)_{\alpha_2 - 1}$ of that section. It was pointed out at the end of 10.6 that $\{x^i_j\}$ also satisfies $(Z_i)_{\alpha_2-1}$. Hence $z^1_{\alpha_2} > 0$, as $R_{\alpha_2\alpha_1} > 0$. Thus if there is a primary α_1 -vector x^1 we have

$$A_{\alpha_2\alpha_2} \quad x^1_{\alpha_2} \quad = \quad z^1_{\alpha_2} > 0.$$

This is not possible by 10.~~4~~2. Theorem 1.

It follows that there is only one primary latent vector associated with 0, and therefore there is one classical canonical submatrix of order two associated with 0.

10.21.

In type (2) we may choose the secondary vector x^1 so that it satisfies $\overline{X}^1$ of section 10.16. When this is done $x^1_{\alpha_2} > 0$. It is worth pointing out that it is possible that

$$c \quad x^2_{\alpha_2} = \quad x^1_{\alpha_2},$$

whence, as $A_{\alpha_2\alpha_2}' \quad x^2_{\alpha_2} = 0$,

we might have obtained $x^1_{\alpha_2} = 0$, from the equation

$$A_{\alpha_2\alpha_2} \quad x^1_{\alpha_2} = - x^2_{\alpha_2} + z^1_{\alpha_2} .$$

In the case $R_{\alpha_j\alpha_i} = 0$, $i, j = 1, \ldots, n$ $i \neq j$ we constructed x^i with as many zeros as possible. This is easily seen from theorem 4. For every positive latent α_i-vector may be expressed as

$$x = \sum_{j=i}^{l} c_j \quad x^j , \quad c_i > 0, \quad c_j \geq 0, \quad j = i+1, \ldots, l,$$

whence $\quad x \geq c_i \; x^i$.

Adopting a similar policy in the present case: $l = 2$ and $R_{\alpha_2\alpha_1} > 0$, we might suppose that if $x^1_{\alpha_2} = c \; x^2_{\alpha_2}$ it might be possible to construct a vector x^1 secondary to x^2 so that X^0 holds, where

$$X^0 : \quad x^1_h > 0 \quad \text{when} \quad R^1_{h\alpha_1} > 0 ,$$
$$\qquad x^1_h = 0 \quad \text{when} \quad R^1_{h\alpha_1} = 0 ,$$

where $R^1_{h\alpha_1}$ is defined exactly as $R_{h\alpha_1}$, viz.

$$R_{h\alpha_1} = \sum r_{i_1 i_2} r_{i_2 i_3} \cdots r_{i_{s-1} i_s}, \quad i_1 = h, \quad i_j = \alpha_1$$

summed over all distinct sets $(i_1, \ldots, i_s)$, except that any sum of terms of $R'_{h\alpha_1}$ which may be written in the form $R_{h\alpha_2} \; R_{\alpha_2\alpha_1}$ is put equal to zero (In particular $R'_{\alpha_2\alpha_1} = 0$). However this ~~is~~ may not ~~so~~ be, if there is a j, such that $R_{j\alpha_2} > 0$. Let h be the smallest integer j such that $R_{j\alpha_2} > 0$. The h-th vector component of x^1 satisfies,

$$A_{hh} \; x_h^1 = - x_h^2 + z_h^1$$

where $x_h^2 > 0$.

If $z_h^1 = 0$, (which, if $\{x_j^i\}, j = 1, \ldots, h$ satisfies X^0 for $j < h$) $R'_{h\alpha_1} = 0$ by the usual argument, implies we have

$$x_h^1 < 0,$$

and the positive partial $\alpha_1 - h$ vector secondary to x^2 is $\{x_j^1 + \beta x_j^2\}, \; j = 1, \ldots, h, \; \beta > 0$. Thus $x_{\alpha_2}^1 + \beta x_{\alpha_2}^2 > 0$ which does not satisfy X^0.

To satisfy X^0 we must have $z_h^1 > 0$, and therefore $R'_{h\alpha_1} > 0$ and also $A_{hh}^{-1} (x_h^2 + z_h^1) > 0$. Of course, these conditions are not sufficient, since we have not yet considered x_j^1 when $j > h$. It does not appear to be profitable to continue this line of investigation. We shall add an example of a matrix which has a vector x^1 satisfying X^0.

$$A = \begin{bmatrix} \cdot & & \\ -1 & \cdot & \\ -2 & -1 & \end{bmatrix}$$

$$\alpha_1 = 1 \quad , \quad \alpha_2 = 2 \quad ,$$

$$R_{21} = 1 \quad , \quad R_{32} = 1 \quad , \quad R_{31} = 2 \, ,$$

but $\quad R'_{21} = 0 \quad , \quad R'_{32} = 1 \quad , \quad R'_{31} = 1 \, .$

Here $\qquad x^2 = \{ \cdot , 1, 1 \} ,$

$$x^1 = \{ 1, \cdot , 1 \} .$$

~~The~~ A vector x^1 satisfying Theorem 9 is

$$x^1 = \{ 1, 1, 2 \} .$$

10.22.

The case $\ell = 3$.

As in the case $\ell = 2$ we shall determine the nature of the classical canonical submatrices of A associated with 0, by finding solutions of equations of the type $A\,x^3 = 0$, $A\,x^1 = -x^2$ etc. In our previous notation the h-th vector component of a typical equation is

$$A_{hh}\, x^1_h = - \; x^2_h + z^1_h \, . \tag{21}$$

This is always soluble if A_{hh} is non-singular, and hence the nature of the classical canonical submatrices will be completely determined by (21) when A_{hh} is singular. This is completely analogous to the previously considered cases of $R_{\alpha_j \alpha_i} = 0$, i, j = 1, ..., ℓ , and the case $\ell = 2$. We shall therefore put $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, but R_{21} for r_{21}, etc. The truth of our subsequent assertions when there are non-singular A_{hh} is very simply established. Where necessary we shall occasionally insert formulae referring to the case with non-singular A_{hh} in square brackets.

It will be interesting to illustrate our results by very simple examples. Often we shall choose A_{11}, A_{22}, A_{33}, to be 1×1 null-matrices. This will bring out the similarity of our results to those of the familiar problem of reduction to classical canonical form of nil-potent triangular matrices (those matrices whose only latent root is zero).

10.23.

We obtain a primary latent $3 - [\alpha_3]$-vector, $x^3 \supset 0$, and $x^2 \supset 0$, a primary vector if $R_{32} = 0$, a vector secondary to x^3 if $R_{32} > 0$. We also obtain a partial $1 - 2$ $[\alpha_1 - \alpha_3 - 1]$ solution $x^1 \supset 0$ of $A x = 0$, if $R_{21} = 0$, and a partial $1 - 2$ solution of $A x^1 = - x^2$ if $R_{21} > 0$. These results are essentially a repetition of the case $\iota = 2$, the first two being obtained by considering the matrix

$$\begin{bmatrix} A_{22} & \\ A_{23} & A_{33} \end{bmatrix}, \text{ the last two by considering } \begin{bmatrix} A_{11} & \\ A_{21} & A_{22} \end{bmatrix}.$$

The vector x^1 may thus be

(α) a primary latent vector,

(β) secondary to x^2,

(γ) secondary to x^3.

Whether x^1 is primary or secondary it is a partial $2 - 2$ $[\alpha_2 - \alpha_3 - 1]$ solution of $A x = 0$. Hence we may replace $\{x^1_j\}$ by $\{x^1_j + d_{12} x^2_j\}$, $j = 1, 2$, and still be left with a partial $1 - 2$ solution of $A x = 0$, or $A x = - x^2$. Similarly we may replace x^2 by $x^2 + d_{23} x^3$ and still be left with a vector satisfying $A\,x = 0$, or $A\,x = -\,x^3$.

We shall start with the vectors x^2, x^3, and the partial 1 - 2 vector $\{x_j^1\}$ derived at the beginning of the section and shall then attempt to find a complete set of latent column (or row) vectors. To do this we shall have to replace x^2 by $x^2 + d_{23}\ x^3$, etc., as above. The 3rd component of the relevant equations is therefore

$$(\alpha)\quad A_{33}\ x_3^1 = z_3^1 + d_{12}\,.\ z_3^2\ ,$$

$$(\beta)\quad A_{33}\ x_3^1 = -\,(\,x_3^2 + d_{23}\ x_3^3\,) + z_3^1 + d_{12}\ z_3^2\ ,$$

$$(\gamma)\quad A_{33}\ x_3^1 = -\ x_3^3 + z_3^1 + d_{12}\ z_3^2\ .$$

We may unite the cases into (E), where

$$(\mathrm{E})\ :\ A_{33}\,x_3^1 = c_{12}\ x_3^2 + c_{13}\ x_3^3 + z_3^1 + d_{12}\ z_3^2\ ,$$

where $c_{12} = -1$ when $R_{21} > 0$,

$c_{12} = 0$ when $R_{21} = 0$.

We similarly have

$$(\mathrm{E}')\qquad y_1^{3'}\,A_{11} = f_{32}\ y_1^{2'} + f_{31}\ y_1^{1'} + w_1^{3'} + g_{32}\,w_1^{2'}$$

where $f_{32} = -1$ when $R_{32} > 0$

$f_{32} = 0$ when $R_{31} = 0$,

and $$w_1^i = -\sum_{j=2}^{3} y_j^{i'}\ A_{j1}\ .$$

Classifying the various possibilities according as R_{21}, R_{31} R_{32}, are zero or positive we obtain seven "types". ($R_{21} > 0$, $R_{31} = 0$, $R_{32} > 0$, being impossible). Four of these yield nothing new, and might be written down immediately by considering the case $\ell = 2$ and the case $R_{ij} = 0$, $i \neq j$, $i,j = 1, 2, 3$. For the sake of completeness, we shall enumerate all types.

From the theory of Chapter 6 we do not expect the solutions x^1, x^2, x^3, to be unique. Hence it is not to be expected that all coefficients of (E) will necessarily be determined by the equations. We shall arbitrarily decide to make as many of c_{12}, c_{13}, d_{12}, x^1_3, etc. equal to zero as possible. When $R_{ji} = 0$, $i \neq j$, $i,j = 1, 2, 3$, this gives the vectors of 10.12.

10.24.

(1) $R_{21} = R_{31} = R_{32} = 0$, $A = \begin{bmatrix} \cdot & & \\ \cdot & \cdot & \\ \cdot & \cdot & \cdot \end{bmatrix}$,

we have $z^1_3 = z^2_3 = x^2_3 = 0$ and therefore (E) is . . . . . .

(E) : $A_{33}\, x^1_3 = c_{13}\, x^3_3$.

Hence $c_{13} = 0$, as $x^3_3 > 0$ is linearly independent of the columns of A_{33}. It follows that $x^1_3 = 0$, is a solution and therefore $x^1 > 0$, satisfies $Ax^1 = 0$.

There are therefore three primary (weakly) positive latent column vectors. A similar argument yields three primary positive latent row vectors. It follows that there are three classical canonical submatrices of order 1 associated with 0.

(2) $R_{21} = R_{31} = 0, \quad R_{32} > 0$.

We have $c_{12} = 0$, $A = \begin{bmatrix} \cdot & & \\ \cdot & \cdot & \\ \cdot & -1 & \cdot \end{bmatrix}$.

and $z^1_3 = 0, \quad z^2_3 > 0$.

Hence (E) : $A_{33}\, x^1_3 = c_{13}\, x^3_3 + d_{12}\, z^2_3$,

and putting $c_{13} = d_{12} = 0$ we obtain $x^1_3 = 0$.

The results for latent row vectors are similar to those of the column vectors of type (4) when 3, 2, 1 is put for 1, 2, 3.

Hence $x^3 > 0$, $x^1 > 0$, are primary latent column vectors, $x^2 > 0$ is secondary to x^3, and $y^1 > 0$, $y^2 > 0$ are primary latent row vectors; $y^3 > 0$, is secondary to y^2.

We deduce that there is one classical canonical submatrix of order 1, and one of order 2, associated with 0.

(3) $R_{21} = 0, \quad R_{31} > 0, \quad R_{32} = 0$

We have $c_{12} = 0$

and $z^1_3 > 0, \quad z^2_3 = x^2_3 = 0$, $A = \begin{bmatrix} \cdot & & \\ \cdot & \cdot & \\ -1 & \cdot & \cdot \end{bmatrix}$.

Thus (E) : $A_{33}\, x^1_3 = c_{13}\, x^3_3 + z^1_3$.

We repeat the procedure outlined for type (2) of $\ell = 2$, and obtain $c_{13} = -1$, on positive normalization of $\{x_j^1\}$ $j = 1,2$. We may again choose $x_3^1 > 0$, as $x_3^3 = 0$ satisfies $A_{33}\, x_3^3 = 0$. Hence we obtain $x^1 \supset 0$.

Thus $x^3 \supset 0$, $x^2 \supset 0$, are primary latent vectors, and $x^1 \supset 0$ is secondary to x^3. In this type the row vectors are completely analogous to the column vectors, whence $y^1 \supset 0$, $y^2 \supset 0$, are primary, and $y^3 \supset 0$ is secondary to y^1.

There is a classical canonical submatrix of order 1, and one of order 2, associated with 0.

(4) $R_{21} > 0$, $R_{31} = 0$, $R_{32} = 0$.

We have $c_{12} = -1$,

and $x_3^2 = z_3^2 = z_3^1 = 0$. , $A = \begin{bmatrix} \cdot & & \\ -1 & \cdot & \\ \cdot & \cdot & \cdot \end{bmatrix}$.

Thus (E): $A_{33}\, x_3^1 = c_{13}\, x_3^3$.

This is soluble only for $c_{13} = 0$ (Cf. type (1)). Putting $c_{13} = 0$, we obtain a solution $x_3^1 = 0$.

For row vectors, the results are similar to those of type (2) for column vectors.

The vectors $x^3 \supset 0$, $x^2 \supset 0$ are primary latent vectors, and $x^1 \supset 0$ is secondary to x^2, while $y^1 \supset 0$, $y^3 \supset 0$ are primary, and $y^2 \supset 0$ is secondary to y^1.

There is a classical canonical submatrix of order 1, and one of order 2, associated with 0.

Types (1) - (4) are an elaboration of the types of $\ell = 2$. Types (5) - (7), on the other hand, introduce some new features.

(5) $R_{21} = 0$, $R_{31} > 0$, $R_{32} > 0$.

We have $c_{12} = 0$,

and $z^1_3 > 0$, $z^2_3 > 0$, $x^2_3 > 0$, $A = \begin{bmatrix} \cdot & & \\ \cdot & \cdot & \\ -1 & -1 & \cdot \end{bmatrix}$.

whence (E) : $A_{33} x^1_3 = c^*_{13} x^3_3 + z^1_3 + d_{12} z^2_3$.

As $R_{32} > 0$, x^2 is secondary to x^3. Hence x^1 is either primary or secondary to x^2. But $R_{21} = 0$, and therefore x^1 is not secondary to x^2. It follows that x^1 is primary, whence

$c_{13} = 0$.

We now have (E) : $A_{33} x^1_3 = z^1_3 + d_{12} z^2_3$.

This equation is soluble as the first $n_3 - 1$ columns of A_{33} (of ~~order~~ n_3 ~~say~~) and the positive vector z^1_3 are linearly independent. By Theorem 1, Corollary 2, $d_{12} < 0$. We have constructed a primary latent vector $\{x^1_1, d_{12} x^2_2, x^1_3\}$.

We note that $d_{12} x^2_2 < 0$ while $x^1_1 > 0$.

Changing our notation we shall denote this vector by $x^1 = \{x^1_1, x^2_2, x^1_3\}$ x^1_j.

This vector is "mixed", that is, it contains both positive and negative elements.

In the notation of Chapter 2, $x^1 \,||\, 0$.

As x^2 is secondary to x^3, every primary latent column vector of A associated with 0 is of the form

$$x = \gamma_1 x^1 + \gamma_3 x^3 = \{\gamma_1 x^1_1, \gamma_1 x^1_2 \;\; \gamma_1 x^1_3 + \gamma_3 x^3_3\}.$$

If $\gamma_1 \neq 0$ (γ_1, γ_3 real), $x^1 \,||\, 0$.

Every complete set of generalized latent column vectors contains two primary vectors, which are linearly independent. Hence every complete set contains at least one mixed latent vector.

We may have appeared to stress this point unduly. But it is this essentially mixed latent vector which makes it impossible in some cases to determine the nature of the classical canonical submatrices associated with 0 when $\ell \geq 4$, solely by inspection of the R - conditions.

We shall give the vectors x^1, x^2, x^3 for our example.

$$x^1 = \{1, -1, \cdot\},$$
$$x^2 = \{\cdot\ \ 1, 1\},$$
$$x^3 = \{\cdot\ \ \cdot\ \ 1\}.$$

Let us consider a matrix with singular A_{hh}.
It is not possible to assert either $x_h^1 \geq 0$, or $x_h^1 \leq 0$ for all ~~the~~ $h > \alpha_2$. Given x^1 and x^3 a primary latent vector $x^1 + \beta x^3$ can be found satisfying X^* below. Changing our notation again we shall denote this vector by x^1.

X^* : $x_h^1 = 0$ when $R_{h\alpha_1} = R_{h\alpha_2} = 0$
(this implies $R_{h\alpha_3} = 0$);
$x_h^1 > 0$ when either $R_{h\alpha_1} > 0$, and $R_{h\alpha_2} = 0$
or $R_{h\alpha_3} > 0$;
$x_h^1 < 0$ when $R_{h\alpha_2} > 0$ and $R_{h\alpha_1} > 0$, (this implies $R_{h\alpha_2} > 0$).
~~R 0, R 0)~~ $R_{h\alpha_1} = R_{h\alpha_3} = 0$.

These possibliities are exclusive but not exhaustive. In the remaining case, $R_{h\alpha_1} > 0$, $R_{h\alpha_2} > 0$, $R_{h\alpha_3} = 0$, we cannot in general find a relation between x_h and 0.

The proof of these assertions is simple, and similar in nature to the proofs of Theorem 3 and 9.

An example is

$$A = \left[\begin{array}{c:c:c:c:c} \cdot & & & & \\ \hdashline \cdot & \cdot & & & \\ \hdashline -2 & -1 & 2 & -1 & \\ -1 & -2 & -1 & 2 & \\ \hdashline -1 & -1 & -3 & \cdot & \cdot \end{array}\right]$$

and $x^3 = \{\cdot , \cdot , \cdot \cdot , 15\}$,

$x^2 = \{\cdot , 3 , 4\ 5 , 13\}$,

while the partial $1 - 3$ vector is $\{x_j^1\} = \{3, \cdot , 5\ 4\}$.

Hence $x^1 = \{15, -18, 1, -10, \gamma\}$,

(on positive normalization) which satisfies X^* if $\gamma > 0$.

We note that $x_3^1 = \{1, -10\} \parallel 0$, $x_2^1 < 0$, $x_1^1 > 0$,

and $x^1 \parallel 0$.

The row vectors for this type are similar to the column vectors of type (6). We have primary vectors $x^3 \supset 0$, $x^1 \parallel 0$, $x^2 \supset 0$, secondary to x^3, and the primary row vectors $y^1 \supset 0$, $y^3 \supset 0$, and $y^2 \supset 0$ secondary to y^1.

There is a classical canonical submatrix of order 1, and one of order 2, associated with 0.

(6) $R_{21} > 0$, $R_{31} > 0$, $R_{32} = 0$,

Here $c_{12} = -1$, $A = \begin{bmatrix} \cdot & & \\ -1 & \cdot & \\ -1 & \cdot & \cdot \end{bmatrix}$.

and $z_3^2 = 0$, $z_3^1 \supset 0$, $x_3^2 = 0$.

Therefore (E) : $A_{33}\ x_3^1 = - d_{23}\ x_3^2 + z_3^1$.

As $R_{21} > 0$, the $1 - 2$ $[\alpha_1 - \alpha_3 - 1]$ vector $\{x_j^1\}$ is a partial solution of $A x = - x^2$. We have put $- d_{23}$ for c_{13} in (E) to emphasize that the term $d_{23}\ x_3^3$ arises, if we attempt to make the vector x^1 secondary to $x^2 + d_{23}\ x^3$. (Cf. 10.23.)

As before (E) is soluble for $d_{23} > 0$. As A_{33} has a strictly positive latent vector associated with 0, we may choose $x^1_3 > 0$. Finally we may change our notation and put x^2 for $x^2 + d_{23} x^3$. It follows that $x^2 \supset 0$, as $d_{23} > 0$.

The latent row vectors are similar to the column vectors of (5). There are primary vectors $x^3 \supset 0$, $x^2 \supset 0$; $x^1 \supset 0$ secondary to $x^2 \supset 0$, associated with 0. The latent row vectors are $y^1 \supset 0$, $y^3 \| 0$, primary vectors, and $y^2 \supset 0$ secondary to y^1.

There is a classical canonical submatrix of order 1, and one of order 2, associated with 0.

(7) $R_{21} > 0$, $R_{31} > 0$, $R_{32} > 0$.

$$A = \begin{bmatrix} \cdot & & \\ -1 & \cdot & \\ \cdot & -1 & \cdot \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} \cdot & & \\ -1 & \cdot & \\ -1 & -1 & \cdot \end{bmatrix}.$$

In this type $c_{12} = -1$

and $z^2_3 \supset 0$, $x^2_3 > 0$.

This is again a special case of 10.16, Theorem 9, and indeed it follows from 6.7, that the vectors $x^3 \supset 0$, $x^2 \supset 0$, $x^1 \supset 0$, obtained by that theorem form a complete set of generalized latent vectors. But it is of interest, in view of later results, to investigate this type a little more.

We may choose the 1 - 2 vector $\{x^j_3\}$ $j = 1, 2$, so that it satisfies $\overline{X}^1_2$ of 10.16 (d). (or Cf. $\ell = 2$, type (2)).

When this is done $z^1_3 > 0$.

Here (E) $A_{33}\, x^1_3 = -x^2_3 - d_{23}\, x^2_3 + z^1_3 + d_{12}\, z^2_3$

where we have again put $-d_{23}$ for c_{13}.

Every vector conformal with A_{33} (of order n_3) is expressible in terms of the first $n_3 - 1$ columns of A_{33}, together with <u>either</u> x^3_3 <u>or</u> z^2_3.

Hence there is an x^1_3 and d_{23} such that

$$(E_1) \qquad A_{33}\, x^1_3 = -x^2_3 - d_{23}\, x^3_3 + z^1_3$$

and an x^1_3, d_{12} such that

$$(E_2) \qquad A_{33}\, x^1_3 = -x^2_3 + z^1_3 + d_{12}\, z^2_3 .$$

~~Of the relative merits of (E_1) and (E_2) we shall have more to say later, when discussing the case $\ell > 3$.~~ We note here that if (E_1) is used to obtain x^1_3, then x^1 is secondary to $x^2 + d_{23}\, x^3$.

Thus a modification of the vector x^2 previously obtained is required. On the other hand (E_2) involves no such modification, but involves a modification of the 1 - 2 vector $\{x^1_j\}$ to $\{x^1_j + d_{12}\, x^2_j\}$ $j = 1, 2.$

As $x^3_3 > 0$, we may choose $x^1_3 > 0$. But the sign of d_{23} when (E_1) is used, or that of d_{12} when (E_2) is used, depends on x^2_3 and z^1_3, or on x^2_3 and z^2_3, and may not be positive. Hence $x^2 + d_{23}\, x^3$, in the case of (E_1) or $\{x^1_1,\ x^1_2 + d_{12}\, x^2_2,\ x^1_3\}$ in the case of (E_2) may be mixed.

We know, however, that there is a set of positive generalized

latent vectors by Theorem 9. These may be obtained analogously to that theorem in the following manner. We return to (E), and choose $d_{23} \geqslant 0$, so that $-x_3^2 - d_{23}\, x_3^3 + z_3^1 \leqq 0$.
It now follows from Theorem 1, corollary 2, that $d_{12} > 0$
(This may be done as $x_3^3 > 0$).) Finally we may choose $x_3^1 > 0$ as we have already remarked.

Thus $\{x^2 + d_{23}\, x^3\} \supset 0$, and $\{x_1^1 \quad x_2^1 + d_{12} x_2^2 \, , \quad x_3^1\} \supset 0$.

We have constructed the positive generalized latent vectors as required, and may denote them by x^1, x^2, x^3; Of course when there are non-singular A_{hh}, the above argument leading to positive vectors is more complicated and resembles the proof of Theorem 9.

The results for latent row vectors are similar to those for the column vectors.

There is a primary latent column vector $x^3 \supset 0$, $x^2 \supset 0$ secondary to x^3, and $x^1 \supset 0$ secondary to x^2. Similarly there is a primary latent row vector $y^1 \supset 0$, $y^2 \supset 0$, secondary to y^1, and $y^3 \supset 0$, secondary to y^2.

It follows that there is one classical canonical submatrix of order 3 associated with the latent root 0 of A.

An example of type (7) is

$$A = \begin{bmatrix} \cdot & & \\ -1 & \cdot & \\ -2 & -1 & \cdot \end{bmatrix}$$

where $x^3 = \{\cdot \quad \cdot \quad 1\}$,

$x^2 = \{\cdot \quad 1 \quad 1\}$ to satisfy $\overline{X}^2$ of 10.16.

The 1 - 2 vector $\{x_j^1\}$ may be chosen as $\{1, \ 1\}$ to satisfy

$\bar{X}_2^1$ of 10.16 (d).

We obtain from (E_1)

$$0 = -1 - d_{23} \cdot 1 + 3$$

whence $d_{23} = 2$.

And therefore with the usual change of notation at this point,

$$x^1 = \{1, 1, 1\} \quad \text{say,}$$
$$x^2 = \{\cdot, 1, 3\}$$
$$x^3 = \{\cdot, \cdot, 1\}.$$

The equation (E_2) yields

$$0 = -1 + 3 + d_{12} \cdot 1$$

whence $d_{12} = -2$.

In this case,

$$x^1 = \{1, -1, \cdot\},$$
$$x^2 = \{\cdot, 1, 1\},$$
$$x^3 = \{\cdot, \cdot, 1\}.$$

In this example the vectors obtained on using (E_1) are positive and satisfy $\bar{X}^i$ of 10.16.

We shall finally emphasize again that the trivial example with which each type is illustrated is merely representative of a much more general state of affairs. Each example may be generalized in two ways without altering its essential nature. Thus the zeros in the diagonal may be replaced by irreducible, singular matrices with positive diagonal elements, and non-positive non-diagonal elements. (Cf. 10.1). Further, an arbitrary number of non-singular A_{hh} may be inserted.

10.25.

The case $\ell > 3$.

When ℓ, the number of singular A_{ii} exceeds three we can in general no longer specify the nature of the generalized latent vectors (and thus the classical canonical submatrices) in terms of

R-functions alone. The reason is that the generalized column latent vectors in the case $\ell = 3$ are not always weakly positive, as we have seen, and hence some of the special features of the theory vanish. In particular, many of our results have depended, directly or indirectly, on Theorem 1. When $\ell > 3$, we may no longer be able to appeal to this theorem, and it will be necessary to assume that the vector $z_{\alpha_i\alpha_i}$, $z_{\alpha_i\alpha_i} \| 0$, is linearly dependent, or linearly independent, as the case may be, of the columns of $A_{\alpha_i\alpha_i}$. Below we give an example of two matrices whose R-conditions are the same (zero elements even coincide), but whose classical canonical submatrices associated with the latent root 0 differ.

$$A = \begin{bmatrix} \cdot & & & \\ \cdot & \cdot & & \\ -1 & -1 & \cdot & \\ -1 & -1 & \cdot & \cdot \end{bmatrix} \qquad B = \begin{bmatrix} \cdot & & & \\ \cdot & \cdot & & \\ -1 & -2 & \cdot & \\ -1 & -1 & \cdot & \cdot \end{bmatrix}$$

For both matrices $R_{21} = R_{43} = 0$, $R_{31} = R_{32} = R_{42} = 1$, $R_{41} = 2$.

But A is of rank 1, and so has three (primary) latent vectors associated with 0, while B is of rank 2, and accordingly there are only two such latent vectors. It is worth remarking that the leading principal submatrices of A and B, are of type $\ell = 3$, (5), which, we have seen, yields a primary latent vector x', $x' \| 0$.

10.26.

We shall now turn to the considerations of complete sets of generalized latent vectors when $\ell > 3$. ~~Before we employ the methods used in the cases $= 1, 2, 3$,~~ We shall prove some theorems about sequences of generalized latent vectors. Part of the first is similar to theorem 8, except that no vector is restricted to be positive.

Theorem 10. Let x^s, $s = 1, \ldots, r$ be a sequence of generalized latent vectors,

$$A\, x^s = -\, x^{s+1}, \quad s = 1, \ldots, r-1,$$
$$A\, x^r = 0.$$

(i) There is an $\alpha_i = \beta_s$ say, such that x^s is a ~~vector~~ β_s- vector.

(ii) $\beta_1 < \beta_2 < \cdots < \beta_r$.

(iii) The partial $\beta_s - \beta_{s+1} - 1$ vector $c\{x^s_j\}$ $(c \geq 0)$ is the partial $\beta_s - \beta_{s+1} - 1$ solution of $A\,x = 0$ and $c\, x^s_{\beta_s} = x_{\beta_s}$ the unique vector satisfying

$$A_{\beta_s \beta_s}\, x_{\beta_s} = 0.$$

(We put conventionally $\beta_{r+1} - 1 = k$.)

By 10.4 x^r is an α_i -vector, for some i, say an β_r- vector.

This proves (i) for $s = r$.

It follows that

$$A_{\beta_r \beta_r}\, x^r_{\beta_r} = 0,$$

and this immediately yields (iii) for $s = r$.

The condition (ii) imposes no restriction on x^r (if we do not assume $r > 1$ at this stage).

Let us now suppose inductively that there are generalized latent vectors x^s, $s = t+1, \ldots, r$, $1 \leq t \leq r$, satisfying (i), (iii) and $\beta_{t+1} < \beta_{t+2} < \cdots < \beta_r$.

Let x^t be an h - vector.
As $A\,x^t = -\,x^{t+1}$, $h \leq \beta_{t+1}$, by Theorem 7.
If $h = \beta_{t+1}$ then $A_{\beta_{t+1}\beta_{t+1}} x^t_{\beta_{t+1}} = -\,x^{t+1}_{\beta_{t+1}}$, but by hypothesis

$$x^{t+1}_{\beta_{t+1}} > 0.$$

and this is impossible, by Theorem 1. It follows that $h < \beta_{t+1}$ and that

$$A_{hh}\,x^t_h = 0\,, \quad x^t_h \neq 0\,. \tag{21}$$

Hence there is an α_i, say β_t, such that $h = \beta_t$. We have proved $\beta_t < \beta_{t+1}$. This is (i) and (ii) for $s = t$, and (iii) follows immediately from (8).
The theorem follows by induction.

10.27.
Theorem 11. Let x^i, $i = 1, \ldots, l$, be a complete set of generalized latent vectors. Let x^i be a β_i -vector, and let the x^i be numbered so that

$$\beta_1 \leq \beta_2 \leq \cdots \leq \beta_l\,.$$

Then $$\beta_i \leq \alpha_i\,.$$

Consider the matrix $A_r : [A_{ij}]$, $i, j = \beta_r, \ldots, k$.
Let us omit zero elements from x^s, $s \geq r$.
putting $$\hat{x}^s = \{x^s_{\beta_r}, \; x^s_{\beta_r+1} \; \ldots \; x^s_k\}$$

It is immediately seen that the x^s, $s = r, \ldots, l$ are linearly independent generalized latent vectors of A_r. Hence A_r has at least $l - r + 1$ zero latent roots; and so it contains at least $l - r + 1$ singular A_{ii}. The theorem follows.

Corollary : $\beta_1 = \alpha_1$.

10.28.

In the case $l = 1, 2, 3$, we constructed generalized latent α_s-vectors x^s. It might be supposed that when $l > 3$ there is always a complete set of generalized latent β_s-vectors, where $\beta_s = \alpha_s$. This is not so.

For let
$$A = \begin{bmatrix} \cdot & & & \\ -1 & \cdot & & \\ \cdot & \cdot & \cdot & \\ \cdot & -1 & -1 & \cdot \end{bmatrix}$$

We have the complete set

$$x^1 = \{1 \ \cdot \ \cdot \ \cdot\},$$
$$x^2 = \{\cdot \ 1 \ \cdot \ \cdot\},$$
$$x^4 = \{\cdot \ \cdot \ \cdot \ 1\},$$

and $w^2 = \{\cdot \ 1 \ -1 \ \cdot\}$, a primary vector,

where x^s, w^s is an s - vector.

By 6,4 every complete set is of the form $\alpha x^1 + \alpha' x^2 + \alpha'' x^4 + \beta w^2$, ~~+ w,~~ $\alpha x^2 + \alpha' x^4$, αx^4 and $\gamma x^4 + \delta w^2$

$\alpha \neq 0$, $\delta \neq 0$.

Hence there is no 3-vector in any complete set.

Incidentally, if we put

$$\mathcal{A}' = (3, 4) \qquad \mathcal{A}'' = (1, 2),$$

so that $\alpha_1' = 3$, $\alpha_2' = 4$, $\alpha_1'' = 1$, $\alpha_2'' = 2$,

We have $R_{\alpha_j' \alpha_i'} > 0$ when $j > i$,

$$R_{\alpha_j'' \alpha_i} = 0,$$

and the vectors

$$x^3 = \{ \cdot \ \cdot \ 1 \ 1 \}$$

$$x^4 = \{ \cdot \ \cdot \ \cdot \ 1 \}$$

satisfy 10.16, Theorem 9; but nevertheless x^3 is not part of any complete set (Cf. 10.17.).

10.29.

<u>Theorem 12.</u> <u>Let the classical canonical matrices of A associated with 0 be 1×1. There is a complete set of (primary) latent vectors x^i, $i = 1, \ldots, l$, associated with 0 such that x^i is an α_i- vector.</u>

Let x^i, $i=1,\ldots, l$ be a complete set. By 6.3 all x^i are primary.

Let x^1, be a β_1-vector.

By 10.27 Theorem 11, Corollary $\beta_1 = \alpha_1$.

Let us now suppose that x^i is an α_i vector $i = 1, \ldots, r$

We shall prove that there is a complete set w^i, $i = 1, \ldots, l$ such that w^i is an α_i-vector for $i = 1, \ldots, r+1$.

Let x^i be an β_i-vector, $i = r, r+1, \ldots, r+s$.

If $s = 0$ then $\beta_{r+1} = \alpha_{r+1}$ by theorem 11 and the result is trivially true for $x^i = w^i$; $i = 1, \ldots, l$.

Suppose $\rho > 0$.

The vector x^i, $i = r, \ldots, r+\rho$, is a partial $\alpha_r - \alpha_{r+1} - 1$ solution of $A x = 0$. But this solution is unique, by 10.6, Theorem 2. Hence

$$x^i_h = c_i \cdot x^r_h , \qquad h = 1, \ldots, \alpha_{r+1} - 1, \text{all}$$

where $c_i \neq 0$.

Let us put $w^i = x^i$, $i = 1, \ldots, r, \ r + \rho + 1, \ldots, l$

$$w^i = x^i - c_i \ x^r , \quad i = r+1, \ldots, r + \rho .$$

Then $A w^i = 0, \quad i = 1, \ldots, l$, ~~[illegible]~~

and the w^i are linearly independent, as the x^i are.

Thus the w^i, $i = 1, \ldots, l$, form a complete set of latent vectors associated with 0.

Also

$$w^i_h = x^i_h - c_i \ x^r_h = 0, \qquad h = 1, \ldots, \alpha_{r+1} - 1$$

for $i = r+1, \ldots, r + \rho$,

and w^i is an α_j-vector for some j (as $Aw^i = 0$).

Hence w^i is a β_i-vector, with $\beta_i = \alpha_j > \alpha_r$, $i = r+1, \ldots, l$.

We may now renumber the w^i, $i = r+1, \ldots, l$,

so that ~~the~~ w^i is a β_i' vector and

$$\beta'_{r+1} \leq \beta'_{r+2} \leq \cdots \leq \beta'_l , \qquad \text{and} \quad \beta'_{r+1} > \alpha_r .$$

By Theorem 11 $\qquad \beta'_{r+1} \leq \alpha_{r+1}$,

whence $\qquad \beta'_{r+1} = \alpha_{r+1}$.

Hence we have constructed l latent vectors w^i, such that w^i is an α_i-vector, for $i = 1, \ldots, r+1$.

By induction we obtain the theorem.

10.30.

<u>Theorem 13.</u> <u>Let $R_{\alpha_p \alpha_q} = 0$, $p,q = s, \ldots, r$,</u>

<u>except that $R_{\alpha_r \alpha_s} > 0$.</u>

<u>Then there is no partial $\alpha_s - \alpha_r$ solution of $A x = 0$.</u>

As $R_{\alpha_r \alpha_s} > 0$, $\alpha_s < \alpha_r$ follows.

Let us put

$$A_{r-1} = [A_{ij}] \qquad i, j = \alpha_s, \dots, \alpha_r - 1,$$

$$A_r = [A_{ij}] \qquad i, j = \alpha_s, \dots, \alpha_r,$$

and denote vectors conformable with A_{r-1} by x, w, etc., those conformable with A_r by $\underline{x}$, $\underline{w}$ .....

As $R_{\alpha_p \alpha_q} = 0$, $p, q = s, \dots, r-1$,

it follows from Theorem 5, that there are $r - s$ linearly independent latent α_i-vectors x^i, $i = s, \dots, r-1$, of A_{r-1} associated with 0 such that

$$x^i_h > 0 \quad \text{if} \quad R_{h \alpha_i} > 0,$$

$$x^i_h = 0 \quad \text{if} \quad R_{h \alpha_i} = 0.$$

In the usual way we put

$$z^i_h = - \sum_{j=1}^{h-1} A_{hj} x_j$$

and deduce

$$z^i_{\alpha_r} = 0 \quad \text{as} \quad R_{\alpha_r \alpha_i} = 0, \; i = s+1, \dots, r-1, \tag{22}$$

$$z^s_{\alpha_r} > 0 \quad \text{as} \quad R_{\alpha_r \alpha_s} > 0. \tag{23}$$

The vectors x^i, $i = s, \dots, r-1$, form a complete set of latent vectors of A_{r-1} associated with 0, as the multiplicity of 0 in A_{r-1} is $r - s$.

Hence if $A_{r-1} w = 0$, where w is an α_s-vector,

$$w = \sum_{i=s}^{r-1} c_i x^i, \qquad c_s \neq 0. \tag{24}$$

Let $A_r \underline{w} = 0$, where $\underline{w} = \{w_j\}$ $j = \alpha_s \ldots \alpha_r$

and $\underline{w}$ is an α_s - vector.

Then $A_{r-1} w = 0$,

and hence (24) holds.

Let us normalize w so that $c_s > 0$.

Putting $v_{\alpha_r} = - \sum_{j=1}^{\alpha_r - 1} A_{\alpha_r j} w_j$

we obtain $v_{\alpha_r} = \sum_{i=s}^{r-1} c_i z^i_{\alpha_r} = c_s z^s_{\alpha_r} > 0$,

by (22), (23), (24).

Thus

$$A_{\alpha_r \alpha_r} w_{\alpha_r} = c_s z^s_{\alpha_r} > 0.$$

This is not possible by theorem 1. The theorem follows.

10.31.

We shall now prove an interesting theorem.

<u>Theorem 14.</u> <u>The classical canonical submatrices of A associated with 0 are 1×1 if and only if</u>

$$\underline{R_{\alpha_i \alpha_j} = 0 , \quad i \neq j, \quad i,j = 1, \ldots, l.}$$

(a) If $R_{\alpha_i \alpha_j} = 0$, $i \neq j$, $i,j, = 1, \ldots, l$, the classical canonical submatrices associated with 0 are 1×1 , by theorem 5.

(b) Suppose $R_{\alpha_i \alpha_j} > 0$, for some i, j.

Let $t = \min (i - j)$ when $R_{\alpha_i \alpha_j} > 0$,

and let r, s, be integers such that $R_{\alpha_r \alpha_s} > 0$

and $r - s = t$.

Then $R_{\alpha_p \alpha_q} = 0$, when $p, q = s, \ldots, r$,

except that $R\alpha_r\alpha_s > 0$.
Hence by theorem 13, there is no partial $\alpha_s - \alpha_r$ solution of $A\,x = 0$.

(c) If the classical canonical submatrices associated with 0 are 1 x 1 there are ℓ primary latent vectors x^i. By theorem 12, we may choose x^i to be an α_i-vector.

Hence $A\,x^s = 0$,

where x^s is an α_s-vector, and therefore $\{x_j^s\}$, $j = \alpha_s, \ldots, \alpha_r$ is a partial $\alpha_s - \alpha_r$ solution of $A\,x = 0$.

But by (b) this is impossible.

Hence if there is some i,j for which $R\alpha_i\alpha_j > 0$ the classical canonical submatrices of A associated with 0 are not all 1 x 1.

The theorem is proved.

10.32.

Biorthonormal sets of latent column and row vectors, and principal idempotent and nilpotent elements of A associated with 0.

Two sets of vectors, $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_\ell$, $\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_\ell$ form biorthonormal sets of vectors, when

$$\underline{y}_i'\,\underline{x}_j = \delta_{ij},$$

where

$$\delta_{ii} = 1, \quad \delta_{ij} = 0, \quad i \neq j.$$

(Cf. 10.12).

We shall denote members of biorthonormal sets as $\underline{x}_i$, $\underline{y}_i$, etc.
When ℓ, the number of singular A_{ii} equals 1, theorem 5 applies. Hence there are α_i-vectors $\underline{x}^i$, $\underline{y}^{i'}$ ($\underline{y}'$ is an

α_1 -vector when $y^1_h = 0, \; h > \alpha_i$),
such that

$$y^{1\prime}_1 x^1_1 = 1 ,$$

and if $E = x^1_1, y^{1\prime}_1 > 0$,

E is the principal idempotent element associated with 0.

When $l = 2$, and $R_{\alpha_2 \alpha_1} = 0$, theorems 5 and 6 also apply.
There are biorthonormal sets of primary latent α_i-vectors, $\underline{x}^i$, $\underline{y}^{i\prime}$, $i = 1, 2$, and the principal idempotent element is positive, as

$$E = \underline{x}^1_1 \; \underline{y}^{1\prime} + \underline{x}^2_2 \; \underline{y}^{2\prime}_2 > 0 .$$

In these cases there are no nilpotent elements as the classical canonical submatrices associated with 0 are $1 * 1$.

10.33.

Suppose $l = 2$, $R_{\alpha_2 \alpha_1} > 0$. We have shown in 10.20 that there is a positive primary latent $\alpha_2(\alpha_1)$ - vector $x^2 (y^{1\prime})$ and a positive secondary $\alpha_1(\alpha_2)$- vector $x^1 \; (y^{2\prime})$.
By positive normalization we obtain

$$y^{1\prime} \; x^1 = 1 .$$

Then it follows that

$$y^{2\prime} x^2 = 1 \quad \text{since} \quad y^{2\prime} x^2_2 = -y^{2\prime} A x^1_1 = y^{1\prime} x^1_1 ,$$

and as
$$x^2_h = 0 \quad , h < \alpha_2 ,$$
$$y^1_h = 0 \quad h > \alpha_1 ,$$
we deduce $y^{1\prime} x^2 = 0$.

In general however

$$y^{2\prime} x^1 = r > 0,$$

and so x^i, y^i $i = 1, 2, \ldots$ do not form biorthonormal sets, unless $r = 0$.

Let us put $\underline{x}^1 = x^1 - r x^2$,

$$\underline{x}^2 = x^2, \quad \underline{y}^1 = y^1, \quad \underline{y}^2 = y^2.$$

Then it is immediately seen that $\underline{x}^i, \underline{y}^i$, $i = 1, 2$, form biorthonormal sets. As $\underline{x}^1$ is secondary to $\underline{x}^2$ ($A\underline{x}^2 = 0$) $\underline{x}^1$, $\underline{x}^2$, (as well as $\underline{y}^1, \underline{y}^2$) form a complete set of generalized latent vectors of A associated with 0. This is a set of the required sort.

However, we may not have $\underline{x}^1_2 > 0$.

Example

$$A = \begin{bmatrix} \cdot & & \\ -2 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$x^1 = \tfrac{1}{\sqrt{6}}\{2, 1, \cdot\}, \quad x^2 = \tfrac{1}{\sqrt{6}}\{\cdot\ 3, 3\},$$
$$y^{1\prime} = \tfrac{1}{\sqrt{6}}[3, \cdot, \cdot], \quad y^{2\prime} = \tfrac{1}{\sqrt{6}}[\cdot\ 1, 1],$$
$$r = y^{2\prime} x^1 = {}^1/_6.$$

Hence $\underline{x}^1 = \tfrac{1}{\sqrt{6}}\{2, \tfrac{1}{2}\ -\tfrac{1}{2}\} \not> 0$.

10.34.

In this section we shall find a necessary and sufficient condition for the existence of sets of positive biorthonormal generalized latent vectors in a special case, when $\ell = 2$, and $R_{\alpha_2\alpha_1} > 0$. ~~in a special case~~

Theorem 15. Let $l = 2$ and $R_{\alpha_2 \alpha_1} > 0$,

while $R_{\alpha_1 h} = 0$, $h \neq \alpha_1$,

$R_{h \alpha_2} = 0$, $h \neq \alpha_2$.

There are biorthonormal complete sets of positive generalized latent vectors if and only if

(i) $R_{\alpha_2 h} R_{h \alpha_1} = 0$, $h = \alpha_1 + 1, \ldots, \alpha_2 - 1$.

(ii) There are vectors $\underline{x}^1 \geq 0$, $\underline{y}^2 \geq 0$, belonging to complete sets such that $\underline{x}^1_{\alpha_2} = 0$, $\underline{y}^2_{\alpha_1} = 0$.

(a) Let x^1, x^2, y^1, y^2, be complete sets of latent vectors satisfying Theorem 9. Any complete sets $\underline{x}^1$, $\underline{x}^2$, $\underline{y}^1$, $\underline{y}^2$, are of the form

$$\underline{x}^2 = x^2 \quad , \quad \underline{x}^1 = x^1 + c\, \underline{x}^2 ,$$

$$\underline{y}^1 = y^1 \quad , \quad \underline{y}^2 = y^2 + d\, \underline{y}^1 ,$$

As $x^2_h = 0$, $h < \alpha_2$,

and $y^1_h = 0$, $h > \alpha_1$,

it follows that

$$\underline{x}^1_h > 0, \quad \text{when} \quad R_{h \alpha_1} > 0 ,$$

$$\underline{x}^1_h = 0, \quad \text{when} \quad R_{h \alpha_1} = 0 ,$$

$$\underline{y}^2_h > 0, \quad \text{when} \quad R_{\alpha_2 h} > 0 ,$$

$$\underline{y}^2_h = 0, \quad \text{when} \quad R_{\alpha_2 h} = 0 ,$$

for $h = \alpha_1 + 1, \ldots, \alpha_2 - 1$.

(b) Hence $\underline{y}^{2\prime}_h \, \underline{x}^1_h = 0$

if and only if $R_{\alpha_2 h} R_{h \alpha_1} = 0$, $h = \alpha_1 + 1, \ldots, \alpha_2 - 1$.

(c) As $\underline{y}^2_{\alpha_2} > 0$, and $\underline{x}^1_{\alpha_2} \geq 0$, when $x^1 \geq 0$,

$$\underline{y}^{2\prime}_{\alpha_2} \, \underline{x}^1_{\alpha_2} = 0$$

if and only if $\quad \underline{x}^1_{\alpha_2} = 0$.

Similarly $\quad y^{2\prime}_{\alpha_1} \underline{x}^1_{\alpha_1} = 0$

if and only if $\quad y^2_{\alpha_1} = 0$

when $\quad y^2 \supset 0$.

(d) In 10.33. we saw that the sets of vectors $\underline{x}^1$, $\underline{x}^2$, and $\underline{y}^1$, $\underline{y}^2$, are biorthogonal if and only if $\underline{y}^{2\prime}\underline{x}^1 = 0$.

But $\quad \underline{y}^{2\prime}\underline{x}^1 = \sum_{h=\alpha_1}^{\alpha_2} y^{2\prime}_h x^1_h$

as
$$x^1_h = x^1_h = x^2_h = 0, \qquad h < \alpha_1,$$
$$y^2_h = y^2_h = y^1_h = 0, \qquad h > \alpha_2.$$

It follows that there are positive biorthonormal sets of generalized latent vectors if and only if
$$y^{2\prime}_h x^1_h = 0, \quad h = \alpha_1 + 1, \ldots, \alpha_2 - 1,$$
and the theorem follows from (b) and (c) .

An example is

$$A = \begin{bmatrix} \cdot & & \\ \cdot & 1 & \\ -1 & \cdot & \cdot \end{bmatrix}, \quad \alpha_1 = 1, \quad \alpha_2 = 3,$$

$$\underline{x}^1 = \{1 \quad \cdot \quad \cdot\}, \qquad \underline{x}^2 = \{\cdot \quad \cdot \quad 1\},$$
$$\underline{y}^{1\prime} = [1 \quad \cdot \quad \cdot], \qquad \underline{y}^{2\prime} = [\cdot \quad \cdot \quad 1].$$

10.35.

In 10.20 (Cf. 10.17, Theorem 9), we proved that if $\ell = 2$, $R_{\alpha_2\alpha_1} > 0$, there were complete sets $x^1 \supset 0$, $x^2 \supset 0$, $y^1 \supset 0$, $y^2 \supset 0$.

As $x_h^1 = 0 \quad , \quad h < \alpha_1,$

$y_h^2 = 0 \quad , \quad h > \alpha_2,$

it follows that $y^{2'} x^1 = 0$, when $y_h^{2'} x_h^1 = 0$,

$h = \alpha_1, \ldots, \alpha_2$.

Thus it might appear that the restriction $R_{\alpha_1 h} = 0$, $h < \alpha_1$, $R_{h\alpha_2} = 0$, $h > \alpha_2$, of Theorem 15 is superfluous. This is not so.

Suppose x^2 is any vector secondary to x^1. In Theorem 9 we proved that there was a β_k such that $w^2 = x^2 + \beta x \supset 0$ if and only if $\beta \geq \beta_k$. If $\beta_k > 0$, $w^2_{\alpha_2} > 0$, and theorem 15 can not be satisfied for a secondary latent α_1-vector. Similar remarks apply to the secondary row α_2-vector.

An example is

$$A = \begin{bmatrix} \cdot & & & \\ \cdot & 1 & & \\ -1 & \cdot & \cdot & \\ \cdot & \cdot & -1 & 1 \end{bmatrix},$$

$x^2 = \{\cdot \quad \cdot \quad 1 \quad 1\}$,

$y^{1'} = [1 \quad \cdot \quad \cdot \quad \cdot]$,

and $x^1 = \{1 \quad \cdot \quad \beta \quad \beta-1\}$,

$y^{2'} = [\gamma \quad \cdot \quad 1 \quad \cdot]$.

Hence if $y^{2'} x^1 = 0$ and $\gamma \geq 0$, $\beta \geq 0$, then $\beta = 0$ and $x^1 = \{1 \quad \cdot \quad \cdot \quad -1\} \not\geq 0$.

10.36.

In Theorem 15, the conditions (a) is a "nought and cross" condition of the type we have had many times before. We may see whether it is satisfied by inspecting the submatrices of A. We only require to know which A_{ii} is singular, and whether $A_{ij} = 0$ or $A_{ij} \subset 0$, $i \neq j$.
On the other hand we can not tell whether ~~whether~~ (b) is satisfied by a glance at the matrix. We shall give an example to show that (b) may be satisfied by A and not by B, when A, B have the same R - conditions.

$$A = \begin{bmatrix} \cdot & \cdot & \cdot \\ -2 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} \cdot & \cdot & \cdot \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix},$$

$$x^2 = \{\cdot \quad 3 \quad 3\}, \qquad \underline{x}^2 = \{\cdot \quad 1 \quad 1\},$$
$$x^1 = \{2 \quad 1 \quad \cdot\}, \qquad \underline{x}^1 = \{1 \quad \cdot \quad \cdot\}.$$

Here
$$x_2^1 = \{1+3\beta \quad 3\beta\} \supset 0,$$
when $x_2^1 \geq 0$.

$$\underline{y}^{1\prime} = [1 \quad \cdot \quad \cdot],$$
$$\underline{y}^{2\prime} = [\cdot \quad 1 \quad 1];$$

Here $x_2^1 = 0$, $y_1^2 = 0$,
and $y^{2\prime} x^1 = 0$.

10.37.

Let us now consider the principal idempotent and nilpotent elements when $\iota = 2$, . $R_{\alpha_2 \alpha_1} > 0$

$$E = \underline{x}^1 \underline{y}^{1\prime} + \underline{x}^2 \underline{y}^{2\prime},$$

$$- N = - AE = \underline{x}^{2} \underline{y}^{1\prime}. \text{ (Cf. 6.9.)}$$

We have

$$E^2 = E ,$$

$$N^2 = 0 .$$

In 10.34 we have seen that there are always biorthonormal sets of generalized latent vectors,

$$\underline{x}^1, \underline{x}^2, \underline{y}^1, \underline{y}^2 \quad \text{with} \quad \underline{x}^2 \supset 0, \quad \underline{y}' \supset 0 .$$

Hence $-N \supset 0$.

But in general $\underline{x}^1 \parallel 0$, $\underline{y}^2 \parallel 0$, and so we cannot assert that $E \supset 0$. When $\underline{x}^1 \supset 0$, $\underline{y}^2 \supset 0$, then of course $E \supset 0$.

An example for which $E \parallel 0$ is

$$A = \begin{bmatrix} \cdot & & \\ -2 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix},$$

$$\underline{x}^1 = 1/12 \{4 \quad 1 \quad -1\}, \qquad \underline{y}^1 = 3\{1 \quad \cdot \quad \cdot\},$$

$$\underline{x}^2 = ½ \{\cdot \quad 1 \quad 1\}, \qquad \underline{y}^2 = \{\cdot \quad 1 \quad 1\},$$

$$E = ¼ \begin{bmatrix} 4 & & \\ 1 & 2 & 2 \\ -1 & 2 & 2 \end{bmatrix} \qquad -N = \frac{3}{2} \begin{bmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \end{bmatrix}.$$

As $\underline{x}^i$ $(y^{i\prime})$ is an α_i-vector $i = 1, 2$, we have from $E = \underline{x}^1 \underline{y}^{1\prime} + \underline{x}^2 \underline{y}^{2\prime}$ that

$$E_{rs} = 0 \quad \text{when} \quad \alpha_j \le r < \alpha_{j+1}, \ \alpha_j < s,$$

$$N_{rs} = 0 \quad \text{when} \quad \alpha_j \le r < \alpha_{j+1} \ \alpha_{j-1} < s,$$

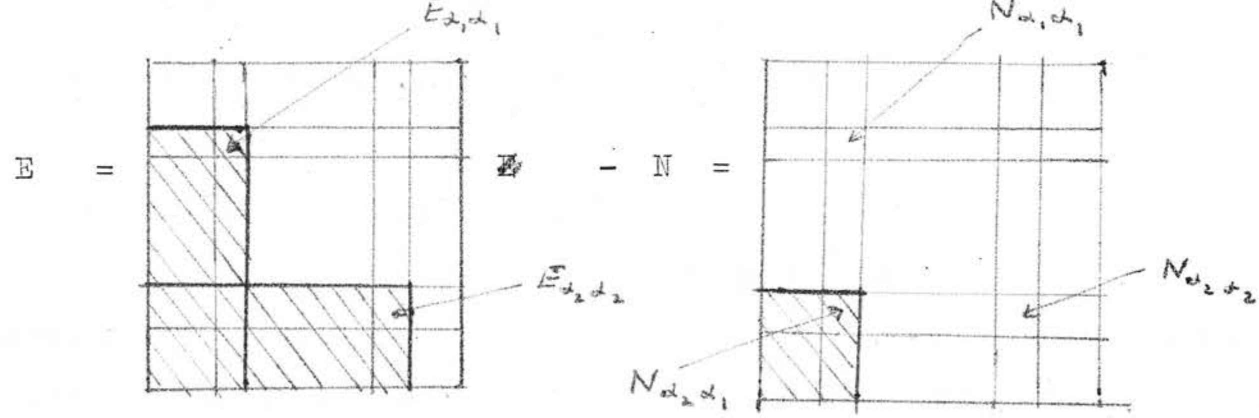
for $j = 0, 1, 2$,

i.e. $N_{rs} \neq 0$ only if $r \geq \alpha_2$, $s \leq \alpha_1$,

where, as before, we have conventionally assumed

$$\alpha_0 = 0 , \quad \alpha_3 = k + 1 .$$

Diagrammatically :



10.38.

In the case $R_{\alpha_2\alpha_1} > 0$, $\ell = 2$, we have already remarked that $E \supset 0$ when there are biorthogonal complete sets of positive latent vectors. The converse is not so obvious.

<u>Theorem 16.</u>

<u>When $\ell = 2$, $R_{\alpha_2\alpha_1} > 0$, then $E \supset 0$, if and only if there are biorthogonal complete sets of positive latent vectors.</u>

We need only prove that there are such sets when $E \supset 0$.

Let $E \supset 0$. There are complete sets of positive generalized latent vectors x^1 , x^2 , y^1 , y^2 . The sets $\underline{x}^1 - c\underline{x}^2$ x^2 and y^1, $y^2 - d\ y^1$ are also complete sets.

Let us choose c_1, d_1 , so that

$$\underline{x}^2 = x^2 \supset 0, \quad \underline{y}^1 = y^1 \supset 0, \quad \underline{x}^1 = x^1 - c_1 x^2 \supset 0,$$

$\underline{y}^2 = y^2 - d_1 y^1 > 0,$

but

$$x^1 - c\, x^2 \parallel 0 \qquad \text{when} \qquad c > c_1 ,$$

$$y^2 - d\, y^1 \parallel 0 \qquad \text{when} \qquad d > d_1 .$$

This is possible as x^1, x^2, and y^1, y^2, are linearly independent.

There is an element of $\underline{x}^1$, ${}_\mu\underline{x}^1$ say, such that ${}_\mu\underline{x}^1 = 0$ but ${}_\mu x^2 > 0$.

Otherwise there would be a $c > c_1$ such that $x^1 - c\, x^2 > 0$.

Similarly there is an element of $\underline{y}^2$, ${}_\nu\underline{y}^2$ say, such that ${}_\nu\underline{y}^2 = 0$, but ${}_\nu\underline{y}^1 > 0$.

In 10.33 we showed that there were biorthogonal sets of generalized latent vectors $\underline{x}^1 - r\, \underline{x}^2$, $\underline{x}^2$, and $\underline{y}^1$, $\underline{y}^2$, where $r \geq 0$.

It follows that

$$E = (\underline{x}^1 - r\, \underline{x}^2)\, \underline{y}^{1\prime} + \underline{x}^2\, \underline{y}^{2\prime}$$

and that μ-ν th element of E is

$${}_\mu\underline{x}^1\, {}_\nu\underline{y}^1 - r\, {}_\mu\underline{x}^2\, {}_\nu\underline{y}^1 + {}_\mu\underline{x}^2\, {}_\nu\underline{y}^2 = -r\, {}_\mu\underline{x}^2\, {}_\nu\underline{y}^1 .$$

But by assumption $E > 0$, and we have shown that

$${}_\mu\underline{x}^2 > 0, \quad {}_\nu\underline{y}^1 > 0.$$

Hence $r = 0$.

It follows that $\underline{x}^1$, $\underline{x}^2$, $\underline{y}^1$, $\underline{y}^2$ are positive biorthogonal complete sets.

This proves the theorem.

Corollary. When $\ell = 2$, $E \supset 0$ if and only if there are complete biorthonormal sets of positive latent vectors.

When $R_{\alpha_2\alpha_1} > 0$ the statement is equivalent to Theorem 16. When $R_{\alpha_2\alpha_1} = 0$, we have seen in 10.13, Theorem 6, that $E \supset 0$, and in 10.12, Theorem 5, that there are complete sets of positive (biorthonormal) latent vectors. The result follows.

10.39.

Let $\ell = 2$, $R_{\alpha_2\alpha_1} > 0$. By 10.33 we may suppose that $\underline{x}^1$, $\underline{x}^2$ and $(\underline{y}^1, \underline{y}^2)$ are complete sets of generalized latent vectors, such that

$$\underline{y}^1 \supset 0, \quad \underline{y}^2 \supset 0, \quad \underline{x}^2 \supset 0, \quad \text{and} \quad \underline{x}^1 + r\,\underline{x}^2 \supset 0$$

for some r.

We have

$$\begin{aligned} E + sN &= \underline{x}^1\,\underline{y}^{1\prime} + \underline{x}^2\,\underline{y}^{2\prime} + s\,\underline{x}^2\,\underline{y}^{1\prime} \\ &= (\underline{x}^1 + s\,\underline{x}^2)\,\underline{y}^{1\prime} + \underline{x}^2\,\underline{y}^{2\prime} \supset 0 \end{aligned}$$

when $s \geq r$.

10.40.

The corollary to theorem 16 has no analogue when $\ell \geq 3$. For suppose

$$A = \begin{bmatrix} \cdot & & \\ \cdot & \cdot & \\ -1 & -1 & \cdot \end{bmatrix}.$$

By 10.24 (5) we know that every complete set of latent column vectors associated with 0 contains one primary latent vector

$$x^i, \quad x^i \,||\, 0 .$$

But as 0 is the only distinct latent root of A,
$Q = [x^1, x^2, x^3]$ is a non-singular 3×3 matrix when x^1, x^2, x^3, form a complete set of latent column vectors. The complete set of latent row vectors, biorthonormal to x^1, x^2, x^3, consists of the rows of Q^{-1}.

Hence

$$E = Q \; Q^{-1} = I \supset 0$$

while the biorthonormal sets of complete vectors are not positive.

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A P P E N D I X

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AN INEQUALITY FOR LATENT ROOTS APPLIED TO DETERMINANTS WITH DOMINANT PRINCIPAL DIAGONAL.

The absolute value of a latent root of a matrix does not exceed the greatest of the sums of absolute values of elements in a row of the matrix. This well-known inequality is due to G. Frobenius (3). In §1 this inequality is generalized by the use of compound matrices. In §2 further inequalities are derived, by means of which bounds for determinants with dominant principal diagonal elements are obtained. These bounds are improvements of bounds due to H. v. Koch (6), and A. Ostrowski (7), which have also been previously improved by A. Ostrowski (7, 8), and G.B. Price (10). In §3 a condition is found under which a matrix is similar to a matrix with dominant principal diagonal when transformed by a diagonal matrix. A distinction is made between singular and non-singular matrices, and it is pointed out that a similar condition of A. Ostrowski (8) may fail in the case of a singular, reducible matrix.

§ 1

Let A be an $n \times n$ matrix with complex elements and C the non-negative matrix $(c_{ij}) = (|a_{ij}|)$. (We use the term non-negative matrix C, positive vector y, etc., to mean a matrix of non-negative elements, a vector of positive elements and write $A \geq 0$, $y > 0$).

Let X be the diagonal matrix $X = \mathrm{diag}\,[x_1, x_2, \ldots, x_n]$ $x_i > 0$, $i = 1, 2, \ldots, n$. The vector r of generalized row sums of A is defined as $r = X^{-1} C X e$, $(r_i = (\sum_{j=1}^{n} |a_{ij}| x_j)/x_i)$, where e is the column vector $e = \{1, 1, \ldots, 1\}$. By R_i, $i = 1, 2, \ldots, n$, we denote an arrangement of the r_i for which $R_1 \geq R_2 \geq \ldots \geq R_n$. The latent roots of A are λ_i, $i = 1, 2, \ldots, n$, supposed arranged so that

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|.$$

We shall use some results on the λ_i and R_i.

$$|\lambda_1| \leq R_1. \tag{1}$$

When A is irreducible $|\lambda_1| = R_1$, if and only if

$$R_1 = R_2 = \quad = R_n \tag{2}$$

$$\text{and} \quad A = e^{i\varphi} D^{-1} C D, \tag{3}$$

where D is a diagonal matrix $|d_{ii}| = 1$, and C is the non-negative matrix defined above. The matrix A is irreducible when it can not be put into the form

$$A = \begin{bmatrix} A_{11} & \cdot \\ & \\ A_{21} & A_{22} \end{bmatrix}$$

by a conjugate permutation of rows and columns, where A_{11}, A_{22} are square matrices and the dot represents a null-matrix.
When A is irreducible, $|\lambda_1| = |\lambda_2| = \quad = |\lambda_k|$ if and only if (2) and (3) hold, and after a conjugate permutation of rows and columns

$$C = \begin{bmatrix} \cdot & C_{12} & & & \\ & & C_{23} & & \\ & & & \ddots & \\ & & & & C_{k-1\,k} \\ C_{k1} & & & & \cdot \end{bmatrix} \qquad (4)$$

the dots in the diagonal denoting square null-matrices.

These conditions were proved by G. Frobenius (3, 4), V. Romanovsky (12) in the non-negative case, A. Brauer (1, 2), H. Wielandt (15), for matrices with complex elements, and others. They are usually stated for $X = I$. For general X they follow from the particular case when $B = X^{-1}A\,X$ is considered, Cf. A. Brauer (1).

When A is irreducible and $|\lambda_1| = R_1$, we give a proof of the necessity of (2), which is rather more concise than those we have found in the literature.

Let u' be a latent row vector associated with λ_1.

$$\lambda_1 u_j = \sum_{i=1}^{n} u_i\, a_{ij} \quad , \quad j = 1, \ldots, n \, .$$

$$|\lambda_1|\,|u_j| \leq \sum_{i=1}^{n} |u_i|\,|a_{ij}| \quad , \quad j = 1, \ldots, n \, , \qquad (5)$$

$$|\lambda_1| \sum_{j=1}^{n} |u_j| \leq \sum_{i,j=1}^{n} |u_i|\,|a_{ij}| = \sum_{i=1}^{n} |u_i|\, r_i \, , (X = I), \qquad (6)$$

$$|\lambda_1| \sum_{j=1}^{n} |u_j| \leq R_1 \sum_{i=1}^{n} |u_i| \, .$$

This proves (1). If $|\lambda_1| = R_1$ we are now able to assert (2) from (6), provided no u_i is zero. Suppose $u_i = 0$, $i = 1, \ldots, k$, $u_i = 0$, $i = k+1, \ldots, n$. The equalities hold in (5) and (6).

By (5) $\quad 0 = R_1 |u_j| = \sum_{i=1}^{k} |u_i|\,|a_{ij}|$, $j = k+1, \ldots, n$

and hence $\quad a_{ij} = 0 \quad i = 1, \ldots, k, \; j = k+1, \ldots, n,$

whence A is reducible. This gives the required condition.

Theorem 1. $\prod_{i=1}^{k} |\lambda_i| \leq \prod_{i=1}^{k} R_i \, , \quad 1 \leq k \leq n \, .$ (7)

Corollary 1. $|\det A| \leq \prod_{i=1}^{n} R_i \, .$ (8)

Corollary 2. When A is irreducible

$$\prod_{i=1}^{k} |\lambda_i| = \prod_{i=1}^{k} R_i \qquad \text{for } k = 1, \ldots, r \leq n, \tag{9}$$

if and only if

$$|\lambda_1| = |\lambda_2| \ldots = \ldots = |\lambda_r| = R_1 = R_2 = \ldots = R_n . \tag{10}$$

Corollary 3. When A is irreducible

$$|\det A| = \prod_{i=1}^{n} R_i \, . \tag{11}$$

if and only if A is monomial or is the 1 x 1 null matrix. (We shall call A monomial if it has precisely one non-zero element in each row and column; cf. Birkhoff and MacLane, "A Survey of Modern Algebra" the MacMillan Company, New York, (1941), p. 227).

Proof of Theorem 1. Let S_γ be a subset of the set 1, 2, .., n containing k members. Let S_γ, $\gamma = 1, \ldots \binom{n}{k}$ form all possible such distinct sets. Let $J_{\mu\gamma}$ be an ordered arrangement of S_γ and $j_{i\mu\gamma}$, $i = 1, \ldots, k$ the i-th member of $J_{\mu\gamma}$. Let $J_{\mu\gamma}$, $\mu = 1, \ldots, k!$ be the possible distinct $J_{\mu\gamma}$ for fixed γ. Evidently as μ and γ run through all values, the $J_{\mu\gamma}$ represent every manner of selecting k distinct integers from n.

Let H_γ be an ordered arrangement of k integers (not necessarily all distinct), from the set 1, ..., n, and let $h_{i\nu}$

be the i-th element of H_ν. Let the H_ν, $\nu = 1, \ldots, n^k$ be all possible distinct H_ν. By $A^{(k)}$ we denote the k-th compound of A, and by $a^{(k)}_{\sigma\tau}$ the element of $A^{(k)}$ formed from the minor at the intersection of rows $i \in S_\sigma$, columns $i \in S_\tau$ of A.
Let $R_i^{(k)}$, $r_i^{(k)}$ be the generalized row sums of $A^{(k)}$.
In the summations below we shall assume that the indices μ, ν, τ run through all the values indicated above. In the product i is taken over the fixed set S_* which we shall choose as 1, ..., k.

We shall first prove the theorem for $X = I$.

$$a^{(k)}_{\sigma\tau} = \sum_{\mu} \pm \prod_{i} a_{ij_{i\mu\tau}}$$

$$c^{(k)}_{\sigma\tau} = \left|a^{(k)}_{\sigma\tau}\right| \leq \sum_{\mu} \prod_{i} c_{ij_{i\mu\tau}}$$

$$r^{(k)}_{\sigma} = \sum_{\tau} c^{(k)}_{\sigma\tau} \leq \sum_{\tau}\sum_{\mu} c_{ij_{i\mu\tau}} \leq \sum_{\nu}\prod_{i} c_{ih_{i\nu}} = \prod_{i} r_i . \tag{12}$$

When $r^{(k)}_{\sigma} = R_1^{(k)}$, we have from (1)

$$\prod_{i=1}^{k} |\lambda_i| \leq r^{(k)}_{\sigma} \leq \prod_{i=1}^{k} r_i \leq \prod_{i=1}^{k} R_i , \tag{13}$$

and (13) may be applied to $B = X^{-1} A X$ to yield the theorem.

Corollary 1, is the case $k = n$, and may be proved more simply by applying (1) to $\phi^{-1} A$, where $\phi = \text{diag}\ [r_1, r_2 \ldots r_n]$.

<u>Proof of Corollary 2.</u> From (9) we have in particular $|\lambda_1| = R_1$. We may now assert (2) and (10) follows. The converse is trivial. In this case A must satisfy (3) and after a conjugate permutation of rows and columns, also (4) with $k = r$.

Proof of Corollary 3. The matrix $A^{(n)}$ consists of the single element $a_{11}^{(n)}$. When $X = I$ we inspect (12), where the equality must hold under the conditions of the corollary. Hence $\sum_{\mu} \prod_{i=1}^{n} c_{i,j_{i\mu}} = \prod_{i=1}^{n} c_{i n_i} = \rho = 1$. As A is irreducible this is satisfied only by a monomial matrix or the 1 x 1 null-matrix. The converse is again evident. For general X we note that det B = det A and that if B is monomial (or null) so is A. This completes the proof. We deduce that when (11) is satisfied by an irreducible A,

$$|\lambda_1| = |\lambda_2| = \qquad = |\lambda_n| = \left(\prod_{i=1}^{n} R_i\right)^{1/n}.$$

Let us now suppose that (9) is satisfied with $r = n$, for an irreducible A.

Then $\quad |\det A| = \prod_{i=1}^{n} R_i .$

By Corollary 3, it follows that A is a monomial matrix or the 1 x 1 null-matrix. If A is monomial $R_i = |a_{ij}| x_j/x_i$, where a_{ij} is the non-zero element in the i-th row of A.
By Corollary 2,

$$R_1 = R_2 = \qquad = R_n = c , \quad \text{say},$$

and hence $C : (c_{ij}), (c_{ij}) = (|a_{ij}|)$, satisfies $C = c\, XPX^{-1}$, where P is a permutation matrix.
By Corollary 2, (3) applies
Thus $\quad A = a\, D^{-1} X P X^{-1} D ,$
where D is a diagonal matrix, $|d_{ii}| = 1$.

If A is reducible we may suppose it decomposed ,

$$A = \begin{bmatrix} A_{11} & \cdot & \cdot & & \cdot \\ A_{21} & A_{22} & & & \cdot \\ \times & \times & \times & & \cdot \\ & & & \times & \cdot \\ A_{r1} & A_{r2} & \times & & A_{rr} \end{bmatrix}, \tag{14}$$

where the A_{ii} are square and irreducible, and the dots represent~~ing~~ null-matrices. The matrix A obtained from A by putting $A_{ij} = 0, \; i > j$, has the same latent roots as A.
Let R_i', $i = 1, \ldots, n$, $R_1' \geq R_2' \geq \ldots \geq R_n' > 0$, be the generalized row sums of A.

Then $R_i' \leq R_i$, $i = 1, \ldots, n$, ~~$= \det A \quad R_i \quad R_i$~~
and by (8) $\prod |\lambda_i| = |\det A| \leq \prod_{i=1}^{n} R_i' \leq \prod_{i=1}^{n} R_i$.
Hence if (11) holds $R_i' = R_i$, and thus $A_{ij} = 0, \; i > j$.
Further restrictions on A are easily found by applying the corollaries to the irreducible A_{ii}.

§ 2.

Let M be a square $n \times n$ matrix whose elements satisfy $m_{ii} \geq 0$, $m_{ij} \leq 0$, $i \neq j$. Let A be a matrix, $(|a_{ij}|) = (|m_{ij}|)$. Let h be the vector $h = X^{-1} M X e$, $e = \{1, 1, \ldots, 1\}$ (i.e. $h_i = 2|a_{ii}| - \sum_{j=1}^{n} (a_{ij} \; x_j) / x_i)$. When $h > 0$, we shall say that A has a dominant diagonal with respect to X (or that A is similar to a matrix with dominant diagonal under transformation X). It was known to Hadamard (5) that A is non-singular when $h > 0$. Many bounds for $|\det A|$ have been found. Recently G.B. Price (10) has given some that are especially simple. Using a method of H.V. Koch (6) we shall deduce bounds which are an improvement on some previously proved.

We shall use a lemma due to H. Weyl (14) and G. Polya (9) to prove further inequalities between the $|\lambda_i|$ and R_i.

Lemma. If $a_i, b_i \geq 0$, $a_1 \geq a_2 \geq \cdots \geq a_n$

and $\prod_{i=1}^{k} a_i \leq \prod_{i=1}^{k} b_i$ $k = 1, \ldots, n$.

Then $\sum_{i=1}^{n} \varphi(a_i) \leq \sum_{i=1}^{n} \varphi(b_i)$,

where $g(\log x) = \varphi(x)$ is a non-decreasing function of $\log x$.

Corollary.
$$\sum_{i=1}^{r} |\lambda_i|^{\alpha} \leq \sum_{i=1}^{r} R_i^{\alpha}, \quad \alpha \geq 0, \quad 1 \leq r \leq n. \tag{15}$$

For $|\lambda_i|$, R_i satisfy the conditions of the lemma by theorem 1, and x^{α} is a non-decreasing function of $\log x$.

Let $h > 0$, $s_i = (1 - h_i/|a_{ii}|)$, $S_1 = \max s_i$, and $S = \sum_{i=1}^{n} s_i$. We have $s_i < 1$, as $h > 0$.

For $X = I$ Ostrowski (7) proved
$$|\det A| \geq \prod_{i=1}^{n} h_i = \prod_{i=1}^{n} |a_{ii}| (1 - s_i) \tag{16}$$
and Koch (6) proved
$$|\det A| \geq e^{S} (1 - S_1)^{S/S_1} \prod_{i=1}^{n} |a_{ii}|. \tag{17}$$

Theorem 2. If $h > 0$
$$e^{S} \prod_{i=1}^{n} |a_{ii}| (1 - s_i) \leq |\det A| \leq \frac{e^{-S} \prod_{i=1}^{n} |a_{ii}|}{\prod_{i=1}^{n} (1 - s_i)} \tag{18}$$

where either inequality holds if, and only if, A is diagonal.

Proof. Let $Q = \text{diag}\,[a_{11} \ldots\ldots a_{nn}]$. Then Q is non-singular as $|a_{ii}| \geq h_i > 0$. Let $P = I - Q^{-1}A$ and let F be the non-negative matrix $(f_{ij}) = (|p_{ij}|)$. The diagonal elements of P are zero and $X^{-1} F X e = s = \{s_1 \ldots\ldots s_n\}$. Let the latent roots of P be μ_i, $i = 1, \ldots, n$.

$$\det A = \det Q \,, \det (I - P) = \prod_{i=1}^{n} a_{ii}\,(1 - \mu_i).$$

Hence $$L = \sum_{i=1}^{n} \log(1 - \mu_i) = -\sum_{i=1}^{n}\sum_{\nu=1}^{\infty} \mu_i^{\nu}/\nu \,,$$

where $L = \log\,(\det A/\prod_{i=1}^{n} a_{ii})$ and the series converges as, by (1), $|\mu_i| \leq s_1 < 1$.

$$L = -\sum_{\nu=2}^{\infty}\frac{1}{\nu}\sum_{i=1}^{n} \mu_i^{\nu} \tag{19}$$

as $\sum_{i=1}^{n}\mu_i = \sum_{i=1}^{n} p_{ii} = 0$,

$$L \leq \sum_{\nu=2}^{\infty}\frac{1}{\nu}\sum_{i=1}^{n}|\mu_i|^{\nu} \tag{20}$$

We apply (15) to P, and obtain from (20)

$$|L| \leq \sum_{\nu=2}^{\infty}\frac{1}{\nu}\sum_{i=1}^{n} s_i^{\nu} = -S + \sum_{i=1}^{n}\sum_{\nu=1}^{\infty}\frac{s_i^{\nu}}{\nu} \,,$$

whence, as $s_i < 1$,

$$|L| \leq -S - \sum_{i=1}^{n} \log\,(1 - s_i)\,. \tag{21}$$

This is equivalent to (18).

One of the equalities in (18) implies the equality in (20). Comparing this equation with (19) we see that μ_i^{ν}, $\nu = 2, 3, \ldots$ have equal arguments. Hence $\mu_i \geq 0$, but $\sum_{i=1}^{n}\mu_i = 0$, and so $\mu_i = 0$, $i = 1, \ldots, n$. This gives $|\det A| = \prod_{i=1}^{n} |a_{ii}|$. We have assumed one of the equalities and thus we must have $s_i = 0$, $i = 1, \ldots, n$. Hence A is diagonal. The converse is obvious.

The lower bound of (18) is evidently as sharp as, or sharper than (16) for all $|\det A|$ for which $h > 0$. That a similar relation holds between (18) and (17) is easily proved : for (17) is equivalent to

$$|L| \leq -S - (S/S_1) \log (1 - S_1),$$

and $- S - (S/S_1)\log (1 - S_1) = - S + (S/S_1) \sum_{\nu=1}^{\infty} \frac{S_1^{\nu}}{\nu}$

$$\geq - S + \sum_{i=1}^{n} \sum_{\nu=1}^{\infty} \frac{s_i^{\nu}}{\nu} = - S - \sum_{i=1}^{n} \log (1 - s_i).$$

The result follows by the equivalence of (21) and (18).

It should be pointed out that other authors have found improvements of (16) and (17) for determinants with a dominant diagonal. Ostrowski (7) has proved

$$|\det A| \geq \prod_{i=1}^{n} |a_{ii}| \quad e^{\sigma^2/S_1} \quad (1 - S_1)^{\sigma^2/S_1}, \tag{22}$$

where $$\sigma^2 = \sum_{i,j=1}^{n} |p_{ij}| \leq (\max|p_{ij}|)\, S \tag{23}$$

and (8)

$$\left(\prod_{i=1}^{n} |a_{ii}|\right) \prod_{i=1}^{[n/2]} (1 - S_{2i-1} S_{2i}) \leq |\det A| \leq \left(\prod_{i=1}^{n} |a_{ii}|\right) \prod_{i=1}^{[n/2]} (1 + S_{2i-1} S_{2i}), \tag{24}$$

where the S_i, $i = 1, \ldots, n$, are an arrangement of the s_i such that $S_1 \geq S_2 \geq \ldots \geq S_n$, and $[n/2]$ is the integral part of $n/2$.

Price (10) has proved

$$\prod_{i=1}^{n} (|a_{ii}| - t_i) \leq |\det A| \leq \prod_{i=1}^{n} (|a_{ii}| + t_i), \tag{25}$$

where $$t_i = \sum_{j=i+1}^{n} |a_{ij}|.$$

For all det A for which $h = X^{-1} M X e > 0$, it is easily seen that (24) and (25), like (18) yield bounds as sharp as, or sharper than (16), while (22) (because of (23)), like (18) yields bounds as sharp as, or sharper than (17). There does not appear to be a relation of this sort between (18), (22), (24), and (25), e.g. (18) may be sharper or less sharp than (22) depending on the particular A considered. One may construct examples to suit one bound or another. Generally speaking, however, (24) gives the best bounds. When the superdiagonal elements of **A** are small compared to the subdiagonal elements (25) is generally best.

§ 3.

If M is a square matrix for which $m_{ii} \geq 0$, $m_{ij} \leq 0$, $i \neq j$, and $h = X^{-1} M X e > 0$ it is known that all latent roots of M have positive real parts, while if $h \geq 0$, the latent roots of M are zero or have positive real parts (Rohrbach (11), Taussky (13)). We shall prove these results with some converses which are apparently new.

Theorem 3. There is an $X = \mathrm{diag}\,[x_1, x_2, \ldots, x_n]$ $x_i > 0$, $i = 1, \ldots, n$ such that $h = X^{-1} M X e > 0$, if and only if the latent roots of $M : (m_{ij})$ $m_{ii} \geq 0$, $m_{ij} \leq 0$, $i \neq j$, have positive real parts.

Among the latent roots of largest modulus of a non-negative matrix G there is one, γ, that is non-negative. If $\alpha > \gamma$, then $(\alpha I - G)$ is non-singular and $(\alpha I - G)^{-1} \geq 0$. (Frobenius (4), Wielandt (15)). Let us choose $\alpha \geq \max m_{ii}$. Then $(\alpha I - M) \geq 0$. We shall now put $G = \alpha I - M$. If $\mu = \alpha - \gamma$ it follows that μ is a latent root of M. Let λ be any other latent root of M, $\lambda \neq \mu$. Then $\beta = \alpha - \lambda$ is a latent root of G. Hence $|\beta| \leq \gamma$ and as $\beta \neq \gamma$, $\mathcal{R}(\beta) < \gamma$ where $\mathcal{R}(\beta)$ is the real part of β.

We have

$$\mathcal{R}(\lambda) = \alpha - \mathcal{R}(\beta) > \alpha - \gamma = \mu .$$

Hence $\underset{m}{\ddot{M}}$ has a latent root, μ, of least real part which is real, and whose real part is less than that of any other latent root.

By applying (1) to G we obtain

$$\gamma = \alpha - \mu \leq \max(X^{-1} \{ \alpha I - M \} \; X e)_i = \alpha - \min h_i ,$$

where $(\dots)_i$ denotes the i-th element of the vector inside the bracket.

Hence $$\min h_i \leqslant \mu \tag{26}$$

for all X . Thus if $h > 0$ it follows that $\mu > 0$, and by $R(\lambda) > \mu$ all latent roots of M have positive real parts.

Suppose that all latent roots have positive real parts. Then M is clearly non-singular and $\mu > 0$. Hence $M^{-1} = (\alpha I - G)^{-1} \geqslant 0$ as $\alpha = \gamma + \mu > \gamma$.

Let k be any column vector, $k > 0$, and let $M X e = k$.

We have
$$Xe = M^{-1} k > 0 .$$

as, of course every row of the non-singular matrix M^{-1} contains at least one positive element. The vector X e determines the diagonal matrix X uniquely. Putting $X^{-1} k = h$, we have $X^{-1} M X e = h > 0$, as required, and the theorem is proved.

While k, $k > 0$, is arbitrary (26) shows that h is much more restricted. It is of some interest to construct an X for which $h > 0$. Let μ_p be the real latent root of least real part of M_{pp}, where the M_{ii} , $i = 1, \dots, r$, are the matrices in the diagonal of the decomposition of M, Cf. (14). Let y^p be a latent vector of M_{pp}, associated with r_p. If G is partitioned conformally with M, we have $G_{pp} = \alpha I_p - M_{pp}$ where I_p is a unit matrix. Then y^p is also a latent column vector of G_{pp} associated with

$\gamma_\rho = \alpha - \mu_p$, where γ_ρ is a non-negative latent root of maximum modulus of G_{pp}. But G_{pp} is irreducible, as an irreducible matrix remains irreducible when the diagonal elements are altered. Hence γ_ρ is a single latent root of G_{pp} and has a unique positive latent column vector associated with it, (Frobenius (4), Wielandt (15)). It follows that $y^\rho > 0$, on suitable normalization.

Let $Y_p = \text{diag}\,(y_1^\rho, y_2^\rho \ldots)$ and $X = \text{diag}\,[\varepsilon_i Y_1 \ldots \varepsilon_r Y_r]$, where the ε_i are positive constants. We shall write $h = \{h^1, \ldots, h^r\}$ h^ρ conformal with $M_{\rho\rho}$. Similarly $e = \{e^1, \ldots, e^r\} = \{1, 1, \ldots, 1\}$. We note that $\mu_\rho \geq \mu > 0$ and $Y_\rho^{-1} M_{pp} Y_p e^\rho = Y_\rho^{-1} \mu_\rho y^\rho = \mu_\rho e^\rho$.

Now $\qquad h = X^{-1} M X e,$

whence $\quad h^\rho = \varepsilon_\rho^{-1} Y_p^{-1} \sum_{j=1}^{\rho} \varepsilon_j M_{pj} Y_j e^j$

$$h^\rho = r_p e^p - \varepsilon_p^{-1} f^p, \tag{27}$$

$$\text{where} \quad f^p = - Y_p^{-1} \sum_{j=1}^{\rho-1} \varepsilon_j M_{pj} Y_j e^j. \tag{28}$$

As $M_{pj} \leq 0$, $j < p$, it follows that $f^p \geq 0$. But f^p, $\rho \geq 2$, is homogeneous and linear in $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{p-1}$ and involves no other ε_i, while $f^1 = 0$. Hence from (27), $h^1 > 0$, and by choosing ε_p sufficiently large compared to $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p$ we shall obtain $h^p > 0$, $p \geq 2$. Choosing successively $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$ we may ensure that $h > 0$.

An implication of the conditions of theorem 3 is worth noting. Let us suppose that m_{ii} is the k1th element of the principal diagonal of M_{pp}. Then for all $M = (m_{ij})$, $m_{ii} \geq 0$, $m_{ij} \leq 0$, $i \neq j$,

$$m_{ii} \geq (Y_p^{-1} M_{pp} Y_p \; e^p)_k \; = \; \mu_p \, ;$$

whence, by $\mu_p = \mu$, $p = 1, \ldots, r$,

$$m_{ii} \geq \mu , \qquad i = 1, \ldots, n.$$

It follows that the equivalent assumptions $h = X^{-1} M X e > 0$ or $\mu > 0$ of theorem 3 imply $m_{ii} > 0$, $i = 1, \ldots, n$.

We shall use the above construction for h to prove a result corresponding to theorem 3 when M is not restricted to be non-singular. Let us say that an irreducible A_{ii} of the decomposition (14) of a reducible matrix A is isolated when $A_{ij} = 0$, $j < i$.

Theorem 4. The matrices M and X satisfy the conditions of theorem 3. If there is an X for which $h = X^{-1} M X e \geq 0$ the latent roots of M are zero or have positive real parts. If the latent roots of M have non-negative real parts there is an X for which $h = X^{-1} M X e \geq 0$ if, and only if, M is irreducible or the singular M_{ii} of the decomposition of M are isolated.

We shall give all symbols the same meanings as in the proof of theorem 3.

The considerations leading to (26) and (27) hold under the conditions of the theorem. If $\mu < 0$ we have from (26) that some $h_i \leq \mu < 0$, for all X . Hence if $h = X^{-1} M X e \geq 0$ then also $\mu \geq 0$, and as $\mathcal{R}(\lambda) > \mu$, when λ is any other latent root of M, $\lambda \neq \mu$, the first part of the theorem follows.

Suppose $\mu \geq 0$ and consider (27).

If $\mu_p > 0$, we have proved that $h' > 0$, and $h^p > 0$, $p \geq 2$, provided that ε_p is sufficiently large compared to $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{p-1}$.

Now suppose $\mu = 0$ and $\mu_p = \mu$.

If M_{pp} is isolated, $M_{pj} = 0$, $j < p$, and by (28), $f^p = 0$, whence $h^p = 0$. Thus if all singular M_{pp} are isolated or if $M = M_{11}$ is irreducible we have a vector $h = \{h^1 \ldots\ldots h^r\}$ where either $h^p > 0$ or $h^p = 0$. Hence $h \geq 0$.

We must still consider the case when some singular M_{pp} is not isolated.

Partition $X = \operatorname{diag}[x_1 \ldots\ldots x_n]$, $x_i > 0$, and $G = \alpha I - M$ conformably with M. Then $X = \operatorname{diag}[X_1, X_2, \ldots, X_r]$ where the X_i are diagonal matrices with positive diagonal elements, and the G_{pp} are irreducible, as an irreducible matrix remains irreducible when the diagonal elements are altered.

We also partition h conformally with M.

$$h = X^{-1} M X e ,$$

$$h^p = X_p^{-1} \sum_{j=1}^{p} M_{pj} X_j e^j ,$$

whence $\quad h^p = \alpha e^p - r^p - f^p ,$

where
$$f^p = - X_p^{-1} \sum_{j=1}^{p-1} M_{pj} X_j e^j , \tag{29}$$

and
$$\alpha e^p - r^p = X_p^{-1} M_{pp} X_p e^p = X_p^{-1} (\alpha I_p - G_p) X_p e^p . \tag{30}$$

Hence
$$r^p = X_p^{-1} G_{pp} X_p e^p . \tag{31}$$

As $M_{pj} \leq 0$, $j = 1, 2, \ldots, p-1$, and some $M_{pj} \neq 0$, $j < p$, it follows from (30) that

$$f^p \geq 0 , \quad f^p \neq 0. \tag{32}$$

But $G_{pp} \geqslant 0$, and so r^p is the vector of generalized row sums of G_{pp}. The latent root of maximum modulus of G_{pp} is $\gamma_p = \alpha - \mu_p = \alpha$. By (1), r_i^p, the largest element of r^p satisfies $r_i^p \geqslant \alpha$.

If $r_i^p > \alpha$, by (29) and (32)

$$h_i^p = \alpha - r_i^p - f_i^p < 0$$

and h is not non-negative.

Suppose $r_i^p = \alpha$. Since G_{pp} is irreducible (2) holds. Hence $r^p = \alpha e^p$, and from (29) and (32)

$$h^p = - f^p \leq 0, \quad h^p \neq 0$$

and h is not non-negative.

As X is an arbitrary diagonal matrix with $x_i > 0$, we have proved that $h = X^{-1} M X e$ is not non-negative when a singular M_{pp} is not isolated. This completes the proof of the theorem.

In theorem 3 and 4 we may replace

$$X = \text{Diag} [x_1, x_2, \ldots, x_n], \; x_i > 0 \tag{33}$$

by

$$x > 0 \tag{34}$$

and

$$h = X^{-1} M X e \geqslant 0 \quad (h = X^{-1} M X e > 0) \tag{35}$$

by

$$k = M x \geqq 0 \quad (k = M x > 0). \tag{36}$$

For putting $Xe = x$ and $X^{-1} h = k$ we have (33) if and only if (34), and (35) if and only if (36).

When $\mu = \alpha - \gamma \geqslant 0$ ($\mu = \alpha - \gamma > 0$) $\det M = \det (\alpha I - G)$ and all principal minors of M are non-negative (positive), (Frobenius (3), Ostrowski (8)). The converse is also true. The characteristic equation of M is

$$\varphi(x) = (-x)^n + c_1(-x)^{n-1} + c_2(-x)^{n-2} + \ldots\ldots + c_n$$

where c_r is the sum of principal minors of M or order r and $c_n = \det M$. If det M is non-negative (positive) and all principal minors of M are non-negative it follows that all real latent roots are non-negative (positive). Hence $\mu \geq 0$ ($\mu > 0$). We deduce that in theorems 3 and 4 we may replace "the latent roots of M are zero or have positive real parts (the latent roots of M have positive real parts)" by "det M and all principal minors are non-negative (positive)" or by "det M is non-negative (positive) and all principal minors of M are non-negative".

A theorem by Ostrowski ((8), theorem 4) may be restated thus: "If in M, $m_{ii} \geq 0$, $m_{ij} \leq 0$, $i \neq j$, det M and all principal minors of M are non-negative there are column vectors, $x > 0$, $k \geq 0$, such that $Mx = k$. If M is non-singular we may restrict $k : k > 0$." Our preceding two remarks indicate that theorem 3 may be reduced to the non-singular case of Ostrowski's theorem. If M is reducible and singular, however, a similar reduction applied to theorem 4 shows that any matrix M with a non-isolated singular M_{ii} in its decomposition will not satisfy Ostrowski's theorem.(There does not appear to be any justification for the second sentence of (8), § 13, page 87.)
An example of such a matrix is

$$A = \begin{bmatrix} \cdot & \cdot \\ -1 & \cdot \end{bmatrix}$$

Finally we shall prove another analogue of Ostrowski's theorem.

Theorem 5. The matrix $M : (m_{ij})$ has $m_{ii} \geq 0$, $m_{ij} \leq 0$, $i \neq j$. If there are vectors $x > 0$, $k \geq 0$, such that $Mx = k$, the latent roots of M have non-negative real parts.

If the latent roots of M have non-negative real parts there are vectors $x \geq 0$, $x \neq 0$, $k \geq 0$, such that $Mx = k$.

The first statement follows from the first part of theorem 4 and the equivalence of (33) and (34), (35) and (36),

Suppose $\mu \geq 0$. Let r be the number of irreducible M_{ii} in the decomposition of M and let $x = \{0^1, \ldots, 0^{r-1}, y^r\}$ where the 0^p are null-vectors and y^r is the positive latent vector of M_{rr} associated with μ_r, the latent root of least real part of M_{rr}.

We have $x \geq 0$, $x \neq 0$,

and
$$k = Mx = \{0^1, \ldots, 0^{r-1}, M_{rr} y^r\}$$
$$= \{0^1, \ldots, 0^{r-1}, \mu_r y^r\}$$
$$\geq 0$$

as $\mu_r \geq \mu \geq 0$.

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